18.06 Problem Set 2

Due Wednesday, 20 February 2008 at 4 pm in 2-106.

Problem 1: a) Do problem 5 from section 2.4 (pg. 65) in the book.
b) Do problem 26 in section 2.4 (pg. 69).

$\boxed{\text{Solution}}$ (5+5 points)

a) We have

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$$

If you insist on a rigorous proof, you can use induction.

b)

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix} \end{aligned}$$

Problem 2: Do problem 24 from section 2.4 (pg. 68).

$\boxed{\text{Solution}}$ (5+5 points)

In general, if we take an upper triangular matrix with 0s along the diagonal, some power of it will be 0. The matrices I picked are all instances of this principle.

a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ will work.

b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Do you see the pattern?

Problem 3: Do problem 7 from section 2.5 (pg. 79).

$\boxed{\text{Solution}}$ (3+4+3 points)

a) Suppose we had a solution x. The equation $Ax = (1, 0, 0)$ amounts to the three equations row $1 \cdot x = 1$, row $2 \cdot x = 0$, row $3 \cdot x = 0$. But we know row $1 +$ row $2 =$ row 3. The three dot products should sum correctly, but they don't:

$$(\text{row } 1 + \text{row } 2) \cdot x = 1 + 0 \neq 0 = \text{row } 3 \cdot x$$

Thus, there can't be a solution x. This shows that A is not invertible.

b) By the same reasoning as above, we must have $b_1 + b_2 = b_3$ for any allowable solution. It's possible that even some of these won't have solutions; we don't have enough information about A to say for sure. For example, the 0 matrix satisfies the requirement to be A, but $0x = b$ certainly won't have any solutions unless all the b_i are 0. We'll learn a more precise way to discuss this when we talk about rank.

c) After elimination row 3 will become all 0. How do we see this? We know from part a) that A is not invertible (if it were, every choice of b would have a a solution). So A can't have three (non-zero) pivots. Since A has three rows but at most two pivots, at least one row must be all 0. (Remember that a pivot is the first non-zero entry in a row; a row without a pivot must not have any non-zero entries at all.) Because we always eliminate downwards, the row without a pivot will be the bottom one (it's possible that the earlier rows also will be all 0).

Another way to do this would be to plug in variables and eliminate by hand.

Problem 4: Define the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$$

Using elimination one can calculate that the inverse is

$$A^{-1} = \begin{bmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

a) Suppose that we formed B by switching the top two rows of A. What would B^{-1} be?

b) Now suppose we defined C by adding three times column(3) of A to column(2). What is C^{-1}?

$\boxed{\text{Solution}}$ (5+5 points)

a) We can describe this operation on A by a matrix P. If

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $B = PA$. Note that P is invertible (it is its own inverse). Thus $B^{-1} = A^{-1}P^{-1}$. Multiplying by $P = P^{-1}$ on the right switches the first two columns, so

$$B^{-1} = \begin{bmatrix} -11 & -16 & 3 \\ \frac{5}{2} & \frac{7}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

Remember, multiplying on the left gives row operations, multiplying on the right gives column operations.

b) We get C by multiplying A on the right by a matrix, $C = AE$ with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

The inverse of E simply replaces 3 by -3. Left multiplication by E^{-1} adds (-3) times row 2 to row 3. Thus, $C^{-1} = E^{-1}A^{-1}$, or

$$C^{-1} = \begin{bmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -13 & -9 & 2 \end{bmatrix}$$

Problem 5: Do problem 23 from section 2.5 (pg. 80).

$\boxed{\text{Solution}}$ (10 points)

$$\left[\begin{array}{cccccc}2&1&0&1&0&0\\1&2&1&0&1&0\\0&1&2&0&0&1\end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc}2&1&0&1&0&0\\0&\frac{3}{2}&1&-\frac{1}{2}&1&0\\0&1&2&0&0&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc}2&1&0&1&0&0\\0&\frac{3}{2}&1&-\frac{1}{2}&1&0\\0&0&\frac{4}{3}&\frac{1}{3}&-\frac{2}{3}&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc}2&1&0&1&0&0\\0&\frac{3}{2}&0&-\frac{3}{4}&\frac{3}{2}&-\frac{3}{4}\\0&0&\frac{4}{3}&\frac{1}{3}&-\frac{2}{3}&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc}2&0&0&\frac{3}{2}&-1&\frac{1}{2}\\0&\frac{3}{2}&0&-\frac{3}{4}&\frac{3}{2}&-\frac{3}{4}\\0&0&\frac{4}{3}&\frac{1}{3}&-\frac{2}{3}&1\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc}1&0&0&\frac{3}{4}&-\frac{1}{2}&\frac{1}{4}\\0&1&0&-\frac{1}{2}&1&-\frac{1}{2}\\0&0&1&\frac{1}{4}&-\frac{1}{2}&\frac{3}{4}\end{array}\right]$$

Problem 6: Do problem 29 from section 2.5 (pg. 81). As with any True/False question, be sure to explain your reasoning: give a brief proof if the statement is true, and give a counterexample if the statement is false.

$\boxed{\text{Solution}}$ (3+3+2+2 points)

a) True. After we eliminate, we will still have a 0 in the pivot position of this particular row, and so A can not be invertible. Alternatively, we can find a system $Ax = b$ with no solutions by putting a 1 in the position of this particular row.

b) False. A counterexample is the matrix consisting of a 1 in every entry.

c) True. To check if A^{-1} is invertible, we need a matrix B so that $A^{-1}B = BA^{-1} = I$. Of course, setting B to be A will work.

d) True. $(A^{-1})^2$ will be the inverse.

Problem 7: Do problem 12 from section 2.6 (pg. 92).

$\boxed{\text{Solution}}$ (10 points)

We can reduce A in one step, using

$$E_{21} = \begin{bmatrix}1&0\\-2&1\end{bmatrix}$$

Then

$$L = E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Now, D is equal to the diagonal of $E_{21}A$, and then U is the rest:

$$\begin{aligned} D &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ U &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We need two steps to reduce B:

$$\begin{aligned} E_{21} &= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ E_{32} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Thus

$$E_{32}E_{21}A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

Now, our L will be

$$L = E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

We pick D to be the diagonal of our upper triangular matrix:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

And U is whatever is left:

$$U = D^{-1}E_{32}E_{21}A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The important thing to note is that U is the tranpose of L for symmetric matrices (that is, U is equal to L flipped over the diagonal).

Problem 8: Do problem 13 in section 2.6 (pg. 93).

$\boxed{\text{Solution}}$ (10 points)

We first reduce the first column. These E_{j1} will consist of a -1 in the correct spot, and the resulting matrix will be

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

The next cancelations will also have a -1 in the appropriate spot, and the resulting matrix will be

$$E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

Similarly the last time:

$$E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

This matrix will be our U, and our L can be computed by taking the inverse of all the E_{ij} and multiplying. As noted in the text, there is a shortcut way for writing down L. You simply take the i,j entry of E_{ij} (the "multiplier"), switch the sign, and put it in the same spot in L. In this case, every E_{ij} has a -1, so

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The conditions on a, b, c, d are $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$. Note that if we factored this into LDU, we would get L and U are transposes just as in the last problem.

Problem 9: Do problem 28 in section 2.6 (pg. 95).

$\boxed{\text{Solution}}$ (10 points)

We reduce A. The first matrix is E_{21} with a -3 multiplier, and as a result we get

$$E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, if $c - 6 = 0$, we will need a row swap here, so $A = LU$ is *impossible*. Furthermore, in this case after a row swap we end up with 3 non-zero pivots. So this is one example answering the question - are there any others?

It's clear that if $c \neq 6$, we won't need to do any row swaps (we're basically done reducing already), and so we will get $A = LU$. Of course, sometimes U will not have 3 pivots (when $c = 7$), but that's not what the question is asking for. LU factorizations will *not* exist only when we need a row swap.

Problem 10: In this problem we will use Matlab to do LU factorizations. Don't forget to include your code! Define the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The command **[L,U]=lu(A)** will decompose A into L and U. We can further decompose U by using the fact that A is symmetric, so that $U = DL'$ (the ' denotes transpose in Matlab). What are L, D, U? What will the pattern be for larger matrices of the same form?

Now, factor **B = [1,2;2,5]** into $B = C'C$ by using **C = chol(B)** (here **chol** stands for Cholesky). Try using the command **[L,U]=lu(B)**. What happens and why? We'll need to include a permutation matrix P via the command **[L,U,P]=lu(B)**. Find L, U, P and check that $PB = LU$.

$\boxed{\text{Solution}}$ (10 points)

My code is below. The pattern in the first part appears to be consecutive ratios $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots$ along the diagonal of D, and their negative inverses inside of L.

Something strange happens when we take the LU decomposition of B: initially the L that we get is not lower-triangular. Even though B does have a legitimate LU-decomposition, Matlab gave us something different. This is because Matlab always rearranges the rows to make pivots as large as possible. When we start out with $A = [1, 2; 2, 5]$, it sees that we can switch the two rows to make the first pivot bigger, and it does that first. This is a good computational technique, but a little confusing if you don't expect it. That is why L is not lower triangular; it includes the row-switch data as well. Including a P will give us a lower triangular matrix, but it's still not the one we would have expected.

```
A = [2,-1,0,0;-1,2,-1,0;0,-1,2,-1;0,0,-1,2]

A =

     2    -1     0     0
    -1     2    -1     0
     0    -1     2    -1
     0     0    -1     2

[L,U]=lu(A)

L =

    1.0000         0         0         0
   -0.5000    1.0000         0         0
         0   -0.6667    1.0000         0
         0         0   -0.7500    1.0000


U =

    2.0000   -1.0000         0         0
         0    1.5000   -1.0000         0
         0         0    1.3333   -1.0000
         0         0         0    1.2500

D = U*(L')^(-1)

D =

    2.0000         0         0         0
         0    1.5000   -0.0000   -0.0000
         0         0    1.3333   -0.0000
         0         0         0    1.2500
```

```
L*D*L'

ans =

    2.0000   -1.0000         0         0
   -1.0000    2.0000   -1.0000   -0.0000
         0   -1.0000    2.0000   -1.0000
         0         0   -1.0000    2.0000

B = [1,2;2,5]

B =

     1     2
     2     5

C = chol(B)

C =

     1     2
     0     1


C'*C

ans =

     1     2
     2     5

[L,U]=lu(B)

L =

    0.5000    1.0000
    1.0000         0
```

```
U =

    2.0000    5.0000
         0   -0.5000

[L,U,P]=lu(B)

L =

    1.0000         0
    0.5000    1.0000


U =

    2.0000    5.0000
         0   -0.5000


P =

     0     1
     1     0

P*B

ans =

     2     5
     1     2

L*U

ans =

     2     5
     1     2

diary off
```