18.06 Problem Set 1

Due Wednesday, 13 February 2008 at 4 pm in 2-106.

Problem 1: Do problem 28 from section 1.1 (pg. 10) in the book.

Solution (10 points)

We are assuming that $abcd \neq 0$, so in particular none of a, b, c, d can be 0.

Now suppose that (a, b) is a multiple of (c, d). That means that for some number k we have $a = kc$ and $b = kd$. Substituting in we find $(a, c) = (kc, c)$ and $(b, d) = (kd, d)$. Of course these are multiples: we get (a, c) by multiplying (b, d) by $\frac{c}{d}$. Note that we must know that d is not 0 in order to be able to define this ratio.

Problem 2: Do problem 31 from section 1.2 (pg. 20).

Solution (10 points)

Yes, three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the plane can have negative dot products with each other. We just need to pick three vectors so that every angle between them is more than 90 degrees. For example, take

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Problem 3: For the system $A\mathbf{x} = \mathbf{b}$ (where A is a 3-by-3 matrix), choose A and $\mathbf{b}$ so that:

1. (row picture) the three planes meet in a common line
2. (row picture) all three planes are parallel but distinct
3. (row picture) the intersection of the first two planes does not intersect the third plane
4. (column picture) $\mathbf{b}$ is not a linear combination of the columns of A

5. (column picture) $\mathbf{b}$ is a multiple of the second column of A

Solution (2+2+2+2+2 points)

1) There are several ways to do this. The easiest is to note that the third equation must be a linear combination of the first two (for example the sum of the first two). One example is

$$\begin{aligned} x &= 0 \\ y &= 0 \\ x + y &= 0 \end{aligned}$$

All three planes contain the z-axis.

Thinking in terms of elimination, what we must do is find a matrix A whose upper triangular form U only has 2 pivots. Of course, this doesn't tell us whether the planes all intersect along a line or whether they don't intersect at all. We need an appropriate choice of $\mathbf{b}$ to put us in the first situation.

2) Each equation must have the same coefficients, but the constant terms must be different. For example

$$\begin{aligned} x &= 0 \\ x &= 1 \\ x &= 2 \end{aligned}$$

3) Similarly to part 1, the third equation must be a linear combination of the first two, but the constant term must then be changed (that is, we need a different $\mathbf{b}$). For example,

$$\begin{aligned} x &= 0 \\ y &= 0 \\ x + y &= 1 \end{aligned}$$

4) The answer to part 3 will also work for this one. All we need is for our system of equations not to have a solution.

5) We can set

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

and take **b** to be the second column.

Problem 4: Do problem 5 from section 2.2 (pg. 42).

Solution (10 points)

Calling the unknown quantity a, we have

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= a \end{aligned}$$

We eliminate by subtracting twice the first row from the second. We find

$$\begin{aligned} 3x + 2y &= 10 \\ 0x + 0y &= a - 20 \end{aligned}$$

If $a = 20$ we have infinitely many solutions, and if a is anything else we will have no solutions. In the case when $a = 20$, every point on the line $3x + 2y = 10$ will be a solution; for example, $(x, y) = (0, 5)$ or $(2, 2)$.

Problem 5: Do problem 12 from section 2.2 (pg. 43).

Solution (10 points)

We have the system

$$\begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

We augment our coefficient matrix and eliminate:

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -3 & 5 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

The bottom equation gives $z = 0$, then the second gives $y = 1$, and finally the top gives $x = 3$.

We needed the steps

$$\begin{aligned}
(-2 * \text{row } 1) + (\text{row } 2) &\mapsto (\text{row } 2) \\
(-1 * \text{row } 1) + (\text{row } 3) &\mapsto (\text{row } 3) \\
(-2 * \text{row } 2) + (\text{row } 3) &\mapsto (\text{row } 3)
\end{aligned}$$

Problem 6: Do problem 19 from section 2.2 (pg. 44).

Solution (5+5 points)

a) Suppose $\mathbf{v} = (x, y, z)$ and $\mathbf{V} = (X, Y, Z)$ are both solutions to $A\mathbf{x} = \mathbf{b}$. Then so is

$$\frac{1}{2}(\mathbf{v} + \mathbf{V}) = \left(\frac{x+X}{2}, \frac{y+Y}{2}, \frac{z+Z}{2}\right)$$

We can check: if $A\mathbf{v} = \mathbf{b}$ and $A\mathbf{V} = \mathbf{b}$ then

$$\begin{aligned}
\frac{1}{2}A(\mathbf{v} + \mathbf{V}) &= \frac{1}{2}(A\mathbf{v} + A\mathbf{V}) \\
&= \frac{1}{2}(\mathbf{b} + \mathbf{b}) \\
&= \mathbf{b}
\end{aligned}$$

Note that $\mathbf{v} + \mathbf{V}$ is not a solution, since it would give $2\mathbf{b}$.

b) By the same reasoning, $t\mathbf{v} + (1-t)\mathbf{V}$ will give a solution for any number t. This means that any vector on the line containing the two points will be a solution. Thus, if 25 planes meet at two points, they meet along the line between the two points.

Another way of thinking about this is to note that if a plane (of any dimension) contains 2 points, it also contains the line between them. Every one of our 25 planes must contain the line between the two points, and so the line is in the intersection of all 25 planes.

Problem 7: Do problem 23 from section 2.3 (pg. 54).

Solution (10 points)

The effect of $E(EA)$ is to do E's operation twice in succession. So, we end up subtracting 4 times row 1 from row 2. Another way of seeing this: $E(EA) = E^2 A$ and $E^2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$.

In opposite order, AE subtracts 2 times column 2 from column 1.

Problem 8: Define the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$. Find a matrix B so that BA is upper triangular. (Hint: first find the elimination matrices for A.)

Solution (10 points)

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 4 & 1 & 8 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

The steps we take are

$$\begin{aligned} (-2 * \text{row } 1) + (\text{row } 2) &\mapsto (\text{row } 2) \\ (-4 * \text{row } 1) + (\text{row } 3) &\mapsto (\text{row } 3) \\ (\text{row } 2) + (\text{row } 3) &\mapsto (\text{row } 3) \end{aligned}$$

So the matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Finally, we take the product (make sure to get the order the right way around!)

$$B = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -6 & 1 & 1 \end{bmatrix}$$

Problem 9: Do problem 27 in section 2.3 (pg. 55)

Solution (10 points)

Setting d will determine whether we are in the "invertible" case (always one solution) or in the "noninvertible" case (either zero solutions or infinitely many). This property only depends on the coefficient matrix A. Both parts ask for the "noninvertible" case, so we must pick d to put ourselves there. So there must be fewer than 3 pivots. Of course this means $d = 0$.

Now, if $c \neq 0$ then there are definitely no solutions. If $c = 0$, there are infinitely many solutions no matter what a and b are. We can choose any number for z and still use back-substitution to determine y and x.

In sum: a and b don't matter, for a) take $d = 0$, $c \neq 0$, for b) take $d = 0$, $c = 0$.

Problem 10: Let's do a warm-up Matlab question. Please include a printout of your Matlab code with your problem set! You can type **diary('filename')** at the beginning of your session to save a transcript, and **diary off** when you're done.

Let's check that in general the products of matrices AB and BA are not equal. (However, as we'll see later, some properties of the two products are the same.) We start with matrices of different sizes. Type in the commands **A=ones(3,2)** and **B=ones(2,3)** (that is, A and B are the 3-by-2 and 2-by-3 matrices with all entries equal to 1). Compute **A*B** and **B*A**. What are their sizes?

Now multiply 3 by 3 matrices C (your choice) and a random D (use the command **D = rand(3,3)**). Does $CD = DC$? Do their diagonals have the same sum (this is called the trace)? Find inv(C) and inv(D).

Solution (10 points)

Here is my code. The trace of the product is the same no matter which one we take first. Note that Matlab returned a warning when I tried to invert C because my choice of C is not invertible:

```
diary('pset1_matlab.txt')
```

```
A = ones(3,2)

A =

     1     1
     1     1
     1     1

B = ones(2,3)

B =

     1     1     1
     1     1     1

A*B

ans =

     2     2     2
     2     2     2
     2     2     2

B*A

ans =

     3     3
     3     3

C = [1,2,3;4,5,6;7,8,9]

C =

     1     2     3
     4     5     6
     7     8     9
```

```
D = rand(3,3)

D =

    0.4447    0.9218    0.4057
    0.6154    0.7382    0.9355
    0.7919    0.1763    0.9169

C*D

ans =

    4.0514    2.9270    5.0274
    9.6076    8.4359   11.8016
   15.1638   13.9447   18.5758

D*C

ans =

    6.9719    8.7441   10.5163
   10.1165   12.4057   14.6948
    7.9153    9.8004   11.6855

trace(C*D)

ans =

   31.0631

trace(D*C)

ans =

   31.0631

inv(C)
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 2.203039e-018.
```

```
ans =

  1.0e+016 *

    0.3152   -0.6304    0.3152
   -0.6304    1.2609   -0.6304
    0.3152   -0.6304    0.3152

inv(D)

ans =

    2.5956   -3.9226    2.8535
    0.8950    0.4383   -0.8432
   -2.4139    3.3037   -1.2119

diary off
```