18.06 Professor Strang Final Exam May 20, 2008

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	`afr`
2)	M 2	4-149	A. Tievsky	2-492	3-4093	`tievsky`
3)	M 3	2-131	A. Ritter	2-085	2-1192	`afr`
4)	M 3	2-132	A. Tievsky	2-492	3-4093	`tievsky`
5)	T 11	2-132	J. Yin	2-333	3-7826	`jbyin`
6)	T 11	8-205	A. Pires	2-251	3-7566	`arita`
7)	T 12	2-132	J. Yin	2-333	3-7826	`jbyin`
8)	T 12	8-205	A. Pires	2-251	3-7566	`arita`
9)	T 12	26-142	P. Buchak	2-093	3-1198	`pmb`
10)	T 1	2-132	B. Lehmann	2-089	3-1195	`lehmann`
11)	T 1	26-142	P. Buchak	2-093	3-1198	`pmb`
12)	T 1	26-168	P. McNamara	2-314	4-1459	`petermc`
13)	T 2	2-132	B. Lehmann	2-089	2-1195	`lehmann`
14)	T 2	26-168	P. McNamara	2-314	4-1459	`petermc`

Thank you for taking 18.06.

If you liked it, you might enjoy 18.085 this fall.

Have a great summer.   GS

1 (10 pts.) The matrix A and the vector b are

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

(a) The complete solution to $Ax = b$ is $x =$ ________.

(b) $A^{\mathrm{T}}y = c$ can be solved for which column vectors $c = (c_1, c_2, c_3, c_4)$? (Asking for conditions on the c's, not just c in $\boldsymbol{C}(A^{\mathrm{T}})$.)

(c) How do those vectors c relate to the special solutions you found in part (a) ?

2 (8 pts.) (a) Suppose $q_1 = (1,1,1,1)/2$ is the first column of Q. How could you find three more columns q_2, q_3, q_4 of Q to make an orthonormal basis? (Not necessary to compute them.)

(b) Suppose that column vector q_1 is an eigenvector of A: $Aq_1 = 3q_1$. (The other columns of Q might not be eigenvectors of A.) Define $T = Q^{-1}AQ$ so that $AQ = QT$. Compare the first columns of AQ and QT to discover *what numbers are in the first column of* T?

3 (12 pts.) Two eigenvalues of this matrix A are $\lambda_1 = 1$ and $\lambda_2 = 2$. The first two pivots are $d_1 = d_2 = 1$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the other eigenvalue λ_3 and the other pivot d_3.

(b) What is the smallest entry a_{33} in the southeast corner that would make A positive semidefinite? What is the smallest c so that $A + cI$ is positive semidefinite?

(c) Starting with one of these vectors $u_0 = (3, 0, 0)$ or $(0, 3, 0)$ or $(0, 0, 3)$, and solving $u_{k+1} = \frac{1}{2} A u_k$, describe the limit behavior of u_k as $k \to \infty$ (with numbers).

4 (10 pts.) Suppose $Ax = b$ has a solution (maybe many solutions). I want to prove two facts:

A. There is a solution x_{row} in the row space $\boldsymbol{C}(A^{\mathrm{T}})$.

B. There is only *one* solution in the row space.

(a) Suppose $Ax = b$. I can split that x into $x_{\text{row}} + x_{\text{null}}$ with x_{null} in the nullspace. How do I know that $Ax_{\text{row}} = b$? (Easy question)

(b) Suppose x^*_{row} is in the row space and $Ax^*_{\text{row}} = b$. I want to prove that x^*_{row} is the same as x_{row}. Their difference $d = x^*_{\text{row}} - x_{\text{row}}$ is in which subspaces? How to prove $d = 0$?

(c) Compute the solution x_{row} in the row space of this matrix A, by solving for c and d:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} x_{\text{row}} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \quad \text{with} \quad x_{\text{row}} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

5 (10 pts.) The numbers D_n satisfy $D_{n+1} = 2D_n - 2D_{n-1}$. This produces a first-order system for $u_n = (D_{n+1}, D_n)$ with this 2 by 2 matrix A:

$$\begin{bmatrix} D_{n+1} \\ D_n \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} \quad \text{or} \quad u_n = Au_{n-1}\,.$$

(a) Find the eigenvalues λ_1, λ_2 of A. Find the eigenvectors x_1, x_2 with second entry equal to 1 so that $x_1 = (z_1, 1)$ and $x_2 = (z_2, 1)$.

(b) What is the inner product of those eigenvectors ? (2 points)

(c) If $u_0 = c_1x_1 + c_2x_2$, give a formula for u_n. For the specific $u_0 = (2, 2)$ find c_1 and c_2 and a formula for D_n.

6 (12 pts.) (a) Suppose q_1, q_2, a_3 are linearly independent, and q_1 and q_2 are already orthonormal. Give a formula for a third orthonormal vector q_3 as a linear combination of q_1, q_2, a_3.

(b) Find the vector q_3 in part (a) when

$$q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(c) Find the projection matrix P onto the subspace spanned by the first two vectors q_1 and q_2. You can give a formula for P using q_1 and q_2 or give a numerical answer.

7 (12 pts.) (a) Find the determinant of this N matrix.

$$N = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

(b) Using the cofactor formula for N^{-1}, tell me one entry that is zero or tell me that all entries of N^{-1} are nonzero.

(c) What is the rank of $N - I$? Find all four eigenvalues of N.

8 (8 pts.) Every invertible matrix A is the product $A = QH$ of an orthogonal matrix Q and a symmetric positive definite matrix H. I will start the proof:

> A has a singular value decomposition $A = U\Sigma V^{\mathrm{T}}$.
> Then $A = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}})$.

(a) Show that UV^{T} is an orthogonal matrix Q (what is the test for an orthogonal matrix ?).

(b) Show that $V\Sigma V^{\mathrm{T}}$ is a symmetric positive definite matrix. What are its eigenvalues and eigenvectors ? Why did I need to assume that A is invertible ?

9 (7 pts.) (a) Find the inverse L^{-1} of this real triangular matrix L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

You can use formulas or Gauss-Jordan elimination or any other method.

(b) Suppose D is the real diagonal matrix $D = \mathsf{diag}(d, d^2, d^3)$. What are the conditions on a and d so that the matrix $A = LDL^{\mathrm{T}}$ is (*three separate questions*, one point each)

(i) invertible? (ii) symmetric? (iii) positive definite?

10 (11 pts.) This problem uses least squares to find the *plane* $C + Dx + Ey = b$ that best fits these 4 points:

$x = 0$	$y = 0$	$b = 2$
$x = 1$	$y = 1$	$b = 1$
$x = 1$	$y = -1$	$b = 0$
$x = -2$	$y = 0$	$b = 1$

(a) Write down 4 equations $Ax = b$ with unknown $x = (C, D, E)$ that would hold if the plane went through the 4 points. Then write down the equations to solve for the best (least squares) solution $\widehat{x} = (\widehat{C}, \widehat{D}, \widehat{E})$.

(b) Find the best $\widehat{x}$ and the error vector e (*is the vector e in $\mathbf{R}^3$ or $\mathbf{R}^4$?*).

(c) If you change this $b = (2, 1, 0, 1)$ to the vector $p = A\widehat{x}$, what will be the best plane to fit these four new points (p_1, p_2, p_3, p_4)? What will be the new error vector?