18.06 Professor Strang Final Exam May 20, 2008

Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	A. Ritter	2-085	2-1192	afr
2)	M 2	4-149	A. Tievsky	2-492	3-4093	tievsky
3)	M 3	2-131	A. Ritter	2-085	2-1192	afr
4)	M 3	2-132	A. Tievsky	2-492	3-4093	tievsky
5)	T 11	2-132	J. Yin	2-333	3-7826	jbyin
6)	T 11	8-205	A. Pires	2-251	3-7566	arita
7)	T 12	2-132	J. Yin	2-333	3-7826	jbyin
8)	T 12	8-205	A. Pires	2-251	3-7566	arita
9)	T 12	26-142	P. Buchak	2-093	3-1198	pmb
10)	T 1	2-132	B. Lehmann	2-089	3-1195	lehmann
11)	T 1	26-142	P. Buchak	2-093	3-1198	pmb
12)	T 1	26-168	P. McNamara	2-314	4-1459	petermc
13)	T 2	2-132	B. Lehmann	2-089	2-1195	lehmann
14)	T 2	26-168	P. McNamara	2-314	4-1459	petermc

Grading

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Thank you for taking 18.06.

If you liked it, you might enjoy 18.085 this fall.

Have a great summer. GS

1 (10 pts.) The matrix A and the vector b are

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

(a) The complete solution to $Ax = b$ is $x =$ ________.

(b) $A^{\mathrm{T}}y = c$ can be solved for which column vectors $c = (c_1, c_2, c_3, c_4)$? (Asking for conditions on the c's, not just c in $\boldsymbol{C}(A^{\mathrm{T}})$.)

(c) How do those vectors c relate to the special solutions you found in part (a)?

Solution (10 points)

a) The complete solution is a particular solution x_p plus any vector in the nullspace x_n. Since the matrix A is already reduced, we can just read the special solutions off: $[-1, 1, 0, 0]^T$ and $[-2, 0, -4, 1]^T$. To find a particular solution to $Ax = b$, we put any numbers (we may as well choose 0) in for the free variables. This yields the two equations $x_1 = 3$ and $x_3 = 1$, so $x_p = [3, 0, 1, 0]^T$. In the end we get

$$x_{comp} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \tag{1}$$

b) You can do this computation by hand by augmenting A^T with the column (c_1, c_2, c_3, c_4) and row reducing. The solution is given by the equations that correspond to 0 rows in the reduced matrix. A quicker way is to note that $A^T y = c$ has a solution whenever c is in the column space $C(A^T)$, i.e. the row space of A. This is perpendicular to the nullspace. Thus, we can find the equations by taking a basis for the nullspace and using the components as coefficients in our equations. We find equations $-c_1 + c_2 = 0$ and $-2c_1 - 4c_3 + c_4 = 0$.

c) Because these c are in the row space, they are perpendicular to vectors in the nullspace of A, and in particular are perpendicular to the special solutions.

2 (8 pts.) (a) Suppose $q_1 = (1,1,1,1)/2$ is the first column of Q. How could you find three more columns q_2, q_3, q_4 of Q to make an orthonormal basis? (Not necessary to compute them.)

(b) Suppose that column vector q_1 is an eigenvector of A: $Aq_1 = 3q_1$. (The other columns of Q might not be eigenvectors of A.) Define $T = Q^{-1}AQ$ so that $AQ = QT$. Compare the first columns of AQ and QT to discover *what numbers are in the first column of T?*

Solution (8 points)

a) First, we find additional vectors v_2, v_3 and v_4 that (along with q_1) make up a basis of $\mathbb{R}^4$. Then we run Gram-Schmidt on q_1, v_2, v_3, v_4.

b) Using the column picture of multiplication, we see that the first column of AQ will be $Aq_1 = 3q_1$. Similarly, if we denote the first column of T by (t_1, t_2, t_3, t_4), then the first column of QT will be $t_1q_1 + t_2q_2 + t_3q_3 + t_4q_4$. Since these two are equal, we get an equality of vectors

$$3q_1 = t_1q_1 + t_2q_2 + t_3q_3 + t_4q_4 \tag{2}$$

Since the q_i are linearly independent, we must have $t_1 = 3$ and the other $t_i = 0$, showing that the first column of T is $(3,0,0,0)$.

We can also note that the first column of T is equal to $3Q^Tq_1$, which yields the same answer.

3 (12 pts.) Two eigenvalues of this matrix A are $\lambda_1 = 1$ and $\lambda_2 = 2$. The first two pivots are $d_1 = d_2 = 1$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the other eigenvalue λ_3 and the other pivot d_3.

(b) What is the smallest entry a_{33} in the southeast corner that would make A positive semidefinite? What is the smallest c so that $A + cI$ is positive semidefinite?

(c) Starting with one of these vectors $u_0 = (3, 0, 0)$ or $(0, 3, 0)$ or $(0, 0, 3)$, and solving $u_{k+1} = \frac{1}{2}Au_k$, describe the limit behavior of u_k as $k \to \infty$ (with numbers).

Solution (12 points)

a) The sum of the eigenvalues is the trace, so $1 + 2 + \lambda_3 = 2$. Thus $\lambda_3 = -1$. The product of the pivots is the determinant, which is the product of the eigenvalues as well. So $d_3 = -2$. Note that this means that A is not positive-definite.

b) We can test positive-definiteness using the determinant method. The two top-left determinants of A are both positive, so we just need to check the third one. We obtain the relation:

$$1(c - 1) + 1(-1) \geq 0 \tag{3}$$

so the smallest value of c is 2.

For the second part, we test whether the eigenvalues are non-negative. The eigenvalues of $A + cI$ are just the eigenvalues of A plus c. So when $c = 1$ all the eigenvalues will be non-negative.

c) The matrix $\frac{1}{2}A$ is a Markov matrix. Because it has some 0 entries, we don't automatically know that it has a unique steady state vector. However, since the eigenvalues of $\frac{1}{2}A$ are $1/2$, $-1/2$ and 1, it does have a unique steady state vector (only one eigenvalue has absolute value 1). To find it, we calculate the eigenvector of A with eigenvalue 2 by taking the nullspace of $A - 2I$:

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \tag{4}$$

$$\rightsquigarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

The nullspace is generated by the special solution $(1, 1, 1)$. So, a vector u will have limit $A^{\infty}u$ equal to $c(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, where c is the sum of the components of u. In particular, the vectors $(3, 0, 0)$, etc., all go to $(1, 1, 1)$.

4 (10 pts.) Suppose $Ax = b$ has a solution (maybe many solutions). I want to prove two facts:

A. There is a solution x_{row} in the row space $\boldsymbol{C}(A^{\mathrm{T}})$.

B. There is only *one* solution in the row space.

(a) Suppose $Ax = b$. I can split that x into $x_{\text{row}} + x_{\text{null}}$ with x_{null} in the nullspace. How do I know that $Ax_{\text{row}} = b$? (Easy question)

(b) Suppose x^*_{row} is in the row space and $Ax^*_{\text{row}} = b$. I want to prove that x^*_{row} is the same as x_{row}. Their difference $d = x^*_{\text{row}} - x_{\text{row}}$ is in which subspaces? How to prove $d = 0$?

(c) Compute the solution x_{row} in the row space of this matrix A, by solving for c and d:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} x_{\text{row}} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \quad \text{with} \quad x_{\text{row}} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Solution (10 points)

a) We have $A(x_{row} + x_{null}) = A(x_{row}) + A(x_{null}) = A(x_{row}) + 0$, so $A(x_{row}) = b$.

b) Suppose both $A(x_{row}) = b$ and $A(x^*_{row}) = b$. Then $x^*_{row} - x_{row}$ is in the row space (since it is a linear combination of vectors in the row space) and is in the nullspace (since multiplying by A will give us 0). Because the row space and nullspace are orthogonal complements, the only vector that is in both is the 0 vector: any vector in both will have $|x|^2 = x \cdot x = 0$.

c) Substituting in the given expressions for $Ax_{row} = b$ we find

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \tag{6}$$

or

$$\begin{bmatrix} 14 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \end{bmatrix} \tag{7}$$

We find $(c, d) = (1, 3)$, so $x_{row} = (4, 5, 0)$. Remark: essentially what we are doing here is projecting onto the row space.

5 (10 pts.) The numbers D_n satisfy $D_{n+1} = 2D_n - 2D_{n-1}$. This produces a first-order system for $u_n = (D_{n+1}, D_n)$ with this 2 by 2 matrix A:

$$\begin{bmatrix} D_{n+1} \\ D_n \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} \quad \text{or} \quad u_n = Au_{n-1}\,.$$

(a) Find the eigenvalues λ_1, λ_2 of A. Find the eigenvectors x_1, x_2 with second entry equal to 1 so that $x_1 = (z_1, 1)$ and $x_2 = (z_2, 1)$.

(b) What is the inner product of those eigenvectors? (2 points)

(c) If $u_0 = c_1x_1 + c_2x_2$, give a formula for u_n. For the specific $u_0 = (2, 2)$ find c_1 and c_2 and a formula for D_n.

Solution (10 points)

a) The eigenvalues of

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \tag{8}$$

satisfy the equation $\lambda^2 - 2\lambda + 2 = 0$, so $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. We find the eigenvectors by taking the appropriate nullspaces:

$$A - \lambda_1 I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \tag{9}$$

has nullspace generated by $x_1 = (1 + i, 1)$, and

$$A - \lambda_2 I = \begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix} \tag{10}$$

has nullspace generated by $x_2 = (1 - i, 1)$. If you pick a different vector in the nullspace, you just rescale so that the bottom entry is 1.

b) The inner product is $x_1^H x_2 = (1-i)^2+1 = 1-2i$, or its conjugate expression $x_2^H x_1 = 1+2i$.

c) If $u_0 = c_1 x_1 + c_2 x_2$, then $u_n = c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2$. A matrix always acts on its eigenvectors in a diagonal way. In particular, $(2, 2) = x_1 + x_2$. So we find

$$u_n = (1+i)^n \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + (1-i)^n \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \tag{11}$$

with second entry $D_n = (1+i)^n + (1-i)^n$.

6 (12 pts.) (a) Suppose q_1, q_2, a_3 are linearly independent, and q_1 and q_2 are already orthonormal. Give a formula for a third orthonormal vector q_3 as a linear combination of q_1, q_2, a_3.

(b) Find the vector q_3 in part (a) when

$$q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(c) Find the projection matrix P onto the subspace spanned by the first two vectors q_1 and q_2. You can give a formula for P using q_1 and q_2 or give a numerical answer.

Solution (12 points)

a) This is the Gram-Schmidt process. We define

$$w_3 = a_3 - (q_1 \cdot a_3)q_1 - (q_2 \cdot a_3)q_2 \tag{12}$$

and then set $q_3 = w_3/\|w_3\|$. Note that we do not need denominators in the expression for w_3 because the q_i are already unit vectors.

b) Substituting in, we find

$$w_3 = a_3 - 5q_1 - (-1)q_2 = (-1, -1, 1, 1) \tag{13}$$

Renormalizing we get $q_3 = \frac{1}{2}(-1, -1, 1, 1)$.

c) The projection matrix P is exactly the expression we used for Gram-Schmidt: $P = q_1q_1^T + q_2q_2^T$. There are other more complicated expressions which are also correct. We can start at the most general and simplify to get this one; if A has columns q_1 and q_2 then $P = A(A^TA)^{-1}A^T = A(I)A^T = q_1q_1^T + q_2q_2^T$ where we used the column-row picture of multiplication for the last step.

7 (12 pts.) (a) Find the determinant of this N matrix.

$$N = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

(b) Using the cofactor formula for N^{-1}, tell me one entry that is zero or tell me that all entries of N^{-1} are nonzero.

(c) What is the rank of $N - I$? Find all four eigenvalues of N.

Solution (12 points)

a) There are many ways to do this. Perhaps the easiest is cofactors along the top row:

$$\det(N) = 1(1) - 4 \det \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix} = 1 - 4(4) = -15 \tag{14}$$

Here I found the determinant of the 3 by 3 by swapping the columns to get an upper triangular matrix with diagonal entries $1, 1, 4$.

b) The cofactor formula is $A^{-1} = C^T / \det(A)$ (we know that A is invertible from part a). To check for 0 entries we can ignore the $\det(A)$ part. We just need to find some cofactors that are 0, and we can arrange this by crossing out rows and columns that will give us a smaller matrix with a column of 0s. Some choices are C_{21}, C_{23}, C_{24}, C_{31}, C_{32}, and C_{34}. The corresponding 0 entries of the inverse are the transposes, so we get the entries $(1, 2)$, $(3, 2)$, $(4, 2)$, $(1, 3)$, $(2, 3)$, $(4, 3)$.

c) The matrix $N - I$ has two columns that are all 0s, and the other two columns are clearly independent, so it has rank 2. So $N - I$ has eigenvalue 0 with multiplicity 2. This tells us that N has eigenvalue 1 with multiplicity 2. Calling the other eigenvalues λ_1 and λ_2, we can find them solving the trace and determinant equations:

$$1 + 1 + \lambda_1 + \lambda_2 = 4 \tag{15}$$

$$(1)(1)\lambda_1\lambda_2 = -15 \tag{16}$$

Thus $\lambda_1 = 5$ and $\lambda_2 = -3$.

8 (8 pts.) Every invertible matrix A is the product $A = QH$ of an orthogonal matrix Q and a symmetric positive definite matrix H. I will start the proof:

> A has a singular value decomposition $A = U\Sigma V^{\mathrm{T}}$.
> Then $A = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}})$.

(a) Show that UV^{T} is an orthogonal matrix Q (what is the test for an orthogonal matrix ?).

(b) Show that $V\Sigma V^{\mathrm{T}}$ is a symmetric positive definite matrix. What are its eigenvalues and eigenvectors ? Why did I need to assume that A is invertible ?

Solution (8 points)

a) To test that $Q = UV^T$ is orthogonal, we must show that $Q^TQ = I$. But $Q^TQ = (UV^T)^TUV^T = VU^TUV^T = V(I)V^T = I$. We used the fact that U and V are orthogonal matrices.

b) The matrix $H = V\Sigma V^T$ is definitely symmetric, as $H^T = V\Sigma^TV^T = V\Sigma V^T$ because Σ is diagonal. Furthermore, note that the expression $H = V\Sigma V^T$ is a diagonalization of H. This means that H has eigenvalues given by the entries of Σ and eigenvectors equal to the columns of V. To show that H is positive-definite, we just need to show that the diagonal entries of Σ are all positive.

Now, we know that they are all non-negative, because the SVD always gives us non-negative singular values. We must also show that none of the singular values are zero. Remember that the singular values are equal to the square roots of the eigenvalues of A^TA. However, because A is invertible, the matrix A^TA is also invertible, and so can't have any eigenvalues equal to 0. So no singular value is 0 either.

9 (7 pts.) (a) Find the inverse L^{-1} of this real triangular matrix L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

You can use formulas or Gauss-Jordan elimination or any other method.

(b) Suppose D is the real diagonal matrix $D = \mathsf{diag}(d, d^2, d^3)$. What are the conditions on a and d so that the matrix $A = LDL^{\mathrm{T}}$ is (*three separate questions*, one point each)

(i) invertible? (ii) symmetric? (iii) positive definite?

Solution (7 points)

a) I'll do Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \tag{17}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & a^2 & -a & 1 \end{array}\right] \tag{18}$$

b) Note that L is invertible no matter what a is, and D is invertible so long as $d \neq 0$. So $A = LDL^T$ will be invertible whenever $d \neq 0$. If $d = 0$, then of course A can't be invertible.
The matrix A is always symmetric, since $A^T = (LDL^T)^T = LD^TL^T = LDL^T$.
Because A is always symmetric, to check positive-definiteness we just need to check that the pivots are all positive. But $A = LDL^T$ is the "pivot" decomposition for A. So the pivots of A are d, d^2, d^3, and we need $d > 0$.

10 (11 pts.) This problem uses least squares to find the *plane* $C + Dx + Ey = b$ that best fits these 4 points:

$$\begin{array}{lll} x = 0 & y = 0 & b = 2 \\ x = 1 & y = 1 & b = 1 \\ x = 1 & y = -1 & b = 0 \\ x = -2 & y = 0 & b = 1 \end{array}$$

(a) Write down 4 equations $Ax = b$ with unknown $x = (C, D, E)$ that would hold if the plane went through the 4 points. Then write down the equations to solve for the best (least squares) solution $\widehat{x} = (\widehat{C}, \widehat{D}, \widehat{E})$.

(b) Find the best $\widehat{x}$ and the error vector e (*is the vector* e *in* $\mathbf{R}^3$ *or* $\mathbf{R}^4$?).

(c) If you change this $b = (2, 1, 0, 1)$ to the vector $p = A\widehat{x}$, what will be the best plane to fit these four new points (p_1, p_2, p_3, p_4) ? What will be the new error vector ?

Solution (11 points)

a) The equations are of the form $C + 0D + 0E = 2$, etc., or in matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad (19)$$

Of course this system is not solvable. The best solution is given by $A^T A\widehat{x} = A^T b$.

b) We have

$$A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{20}$$

and

$$A^T b = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \tag{21}$$

It is a diagonal system, so we immediately find $(C, D, E) = (1, -1/6, 1/2)$. The error vector is the difference of the real b and the approximate values we get for our plane: $e = b - A\widehat{x}$. Since $A\widehat{x} = [1, 4/3, 1/3, 4/3]^T$, we get $e = (1, -1/3, -1/3, -1/3)$.

c) We know $p = A\widehat{x}$ is the projection of b onto the column space of A. So the system $Ax = p$ is solvable exactly; we don't need any approximations. The best fit plane will be the same plane as in part b: $1 - x/6 + y/2 = b$ (we changed the b-coordinates of the points so that they lie on this plane, so of course it is the best fit). The error vector will become 0 because it is an exact fit.