18.06 Problem Set 3 - Solutions

Due Wednesday, Feb. 28, 2007 at **4:00 p.m.** in 2-106

Each problem is worth 10 points. The date next to the problem number indicates the lecture in which the material is covered.

Problem 1 *Tuesday 2/20*

For each of the following questions, please explain your answer.

(a) Let $F = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq y \geq z \geq 0\}$. Is F a subspace of $\mathbb{R}^3$?

(b) The set of all real functions forms a vector space. Is the set of all functions of the form $f(x) = ax^2$ (where a can take any real value) a subspace? How about the set of all functions of the form $f(x) = x^2 + bx + c$ (where b and c can take any real value)?

(c) Let A be a fixed 3×2 matrix. Let F be the set of all 3×3 matrices B such that $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Is F a subspace of the space of all 3×3 matrices?

Solution 1

(a) No, since if we multiply a vector from F by a negative scalar, the result will not anymore be in F.

(b) For ax^2 the answer is yes, since for any two functions f, g of the given form, $f(x) = ax^2$ and $g(x) = bx^2$ the sum $(f + g)(x) = (a + b)x^2$ and the result of multiplying f by a scalar c $cf(x) = (ca)x^2$ are still of the form scalar times x^2.

For $x^2 + bx + c$ the answer is no, since for two functions f, g of the given form, $f(x) = x^2 + b_1x + c_1$ and $g(x) = x^2 + b_2x + c_2$ the sum $(f + g)(x) = 2x^2 + (b_1 + b_2)x + (c_1 + c_2)$ and here x^2 has coefficient 2 as opposed to 1.

(c) Yes, it is a subspace since if $B_1, B_2 \in F$, then we have $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = B_1A + B_2A = (B_1 + B_2)A$ and for any scalar c if $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ then so is $cBA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 2 *Tuesday 2/20*

Do problem 22 of section 3.1 in your book.

Solution 2

There are three systems in the problem. We can obviously solve the first system for any (b_1, b_2, b_3). The second system can be solved provided $b_3 = 0$ since the last equation read $0x_1 + 0x_2 + 0x_3 = b_3$. Finally, in the third system we must have $b_2 = b_3$ since the second and third equations read $x_3 = b_2$ and $x_3 = b_3$.

Problem 3 *Wednesday 2/21*

Do problems 5, 6 and 7 of section 3.2 in your book.

Solution 3

Problem 5. (a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$. The corresponding $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$. The corresponding $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.
Problem 6. The free variables corresponding to A are x_2 and x_3. Therefore, we have two special solutions: $x_2 = 0$, $x_3 = 1$, $x_1 = 5$ and $x_2 = 1$, $x_3 = 0$, $x_1 = 3$.
There is only one free variable corresponding to B, namely x_2. Therefore, we have one special solution: $x_2 = 1$, $x_3 = 0$, $x_1 = 3$.
For an m by n matrix the number of pivot variables plus the number of free variables is n, the number of columns.
Problem 7.
$N(A) = \{c_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} : c_1, c_2 \text{ any scalars}\} = \{\begin{bmatrix} 5c_1 + 3c_2 \\ c_2 \\ c_1 \end{bmatrix} : c_1, c_2 \text{ any scalars}\}$
The equation describing $N(A)$ is $x = 3y + 5z$.
$N(B) = \{c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} : c_1 \text{ any scalar}\} = \{\begin{bmatrix} 3c_1 \\ c_1 \\ 0 \end{bmatrix} : c_1 \text{ any scalar}\}$
The equation describing $N(B)$ is $x = 3y$.

Problem 4 *Wednesday 2/21*

Do problem 25 of section 3.2 in your book.

Solution 4

A possible answer is: $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Since it has 3 pivots we know there is only one free variable, so the nullspace is exactly all the multiples of $(1, 1, 1, 1)$ and nothing more.

Problem 5 *Friday 2/23*

Do problem 7 of section 3.4 in your book.

Solution 5

We have $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}$ and then if we augment it by the vector (b_1, b_2, b_3) and do elimination we get the matrix $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + 4b_1 \end{bmatrix}$. Since the system is then solvable exactly when $b_3 - 2b_2 + 4b_1 = 0$ this implies that (b_1, b_2, b_3) is in the column space exactly when $b_3 - 2b_2 + 4b_1 = 0$. From this equation we also read of that $r_3 - 2r_2 + 4r_1 = 0$, where r_i denotes the i-th row.

Problem 6 *Friday 2/23*

Do problem 21 of section 3.4 in your book.

Solution 6

(a) In the system $x + y + z = 4$ the free variables are y and z, therefore the particular solution is $x_p = (4, 0, 0)$ and the two special solutions are $(-1, 1, 0)$ and $(-1, 0, 1)$. Thus the complete solution is of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(b) By Gaussian elimination the system reduces to $x+y+z=4$ and $-2y=0$ so the free variable is z. The particular solution is then $x_p=(4,0,0)$ and the special solution is $x_n=(-1,0,1)$ and the complete solution is $\begin{bmatrix}4\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Problem 7 *Friday 2/23*

Let $A=\begin{bmatrix}1&2&-2&1&0\\2&4&-3&3&0\\-3&-6&5&4&2\\5&10&-9&6&0\end{bmatrix}$.

(a) Transform A to (ordinary) echelon form.
(b) What are the pivots? What are the free variables?
(c) Now transform A to row reduced echelon form.
(d) Give the special solutions. What is the nullspace $N(A)$?
(e) What is the rank of A?
(f) Give the complete solution to $Ax=b$, where $b=A\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$.

Solution 7

(a) The echelon form is $\begin{bmatrix}1&2&-2&1&0\\0&0&1&1&0\\0&0&0&8&2\\0&0&0&0&0\end{bmatrix}$
(b) The pivots are the $(1,1)$, $(2,3)$ and $(3,4)$ entries, and so the pivot variables are x_1, x_3 and x_4 leaving x_2, x_5 as the free variables.
(c)We get $\begin{bmatrix}1&2&0,0,-0.75\\0&0&1,0,-0.25\\0&0&0,1,0.25\\0&0&0,0,0\end{bmatrix}$.
(d) The special solutions are $\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0.75\\0\\0.25\\-0.25\\1\end{bmatrix}$. Thus, the vectors in $N(A)$ are of the form $c_1\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0.75\\0\\0.25\\-0.25\\1\end{bmatrix}$ for arbitrary scalars c_1 and c_2.
(e) It is the number of pivots, 3.
(f) We already have the special solutions, and so we only need to find a particular solution, which we are already given, namely $\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$. It is important to note that any particular solution works.
We could also get a particular solution by setting the free variables x_2 and x_5 to 0. In this case we can find the particular solution by inspection, since we know that $A\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}=b$ and $A\begin{bmatrix}0.75\\0\\0.25\\-0.25\\1\end{bmatrix}=0$, thus $A(\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}-\begin{bmatrix}0.75\\0\\0.25\\-0.25\\1\end{bmatrix})=b$ so a particular solution to the system is $\begin{bmatrix}0.25\\0\\0.75\\0.25\\0\end{bmatrix}$. Therefore the complete

solution is of the form $\begin{bmatrix} 0.25 \\ 0 \\ 0.75 \\ 0.25 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix}$.

Problem 8 *Monday 2/26*

Suppose the $m \times n$ matrix R is in row reduced echelon form $\begin{matrix} I & F \\ \mathbf{0} & \mathbf{0} \end{matrix}$, with r nonzero rows and first r columns as pivot columns.
(a) Describe the column space and the nullspace of R.
(b) Do the same for the $m \times 2n$ matrix B=$[R \quad R]$.
(c) Do the same for the $2m \times n$ matrix C= $\begin{matrix} R \\ R \end{matrix}$.
(d) Do the same for the $2m \times 2n$ matrix D= $\begin{matrix} R & R \\ R & R \end{matrix}$.

Solution 8

(a) The column space of R consists of all m-dimensional vectors such that the last $m-r$ coordinates are 0.
To get the nullspace, notice that there are exactly r pivot variables corresponding to the first r columns, and there are $n-r$ free variables. We must find $n-r$ special solutions by setting all but one of the free variables to 0 and one to 1. then our nullspace is going to be the span of these special solutions.
(b)Clearly, $C(B) = C(R)$.
The only difference from A for the nullspace of B is that now we have r pivot variables corresponding to the first r columns, and there are $2n-r$ free variables.
(c) The column space of C consists of all $2m$-dimensional vectors such that $r+1$, $r+2$, $\ldots$, mth and the last $m-r$ coordinates are 0.
The nullspace of C is the same as the nullspace of R since we only have the equations repeated here, which does not affect the solutions to the system.
(d) The column space of D consists of all $2m$-dimensional vectors such that the $r+1$, $r+2$, $\ldots$, mth and the last $m-r$ coordinates are 0.
The nullspace of C is the same as the nullspace of A since we only have the equations repeated here, which does not affect the solutions to the system.

Problem 9 *Monday 2/26*

Do problem 17 of section 3.3 in your book.

Solution 9

(a) Call the i^{th} column of B, B_i. Then the problem states that $B_j = c_1B_1 + \ldots + c_{j-1}B_{j-1}$. Let the columns of AB be denoted by $(AB)_i$. It is clear that $(AB)_j = A \cdot B_j = A \cdot (c_1B_1 + \ldots + c_{j-1}B_{j-1}) = c_1A \cdot B_1 + \ldots + c_{j-1}A \cdot B_{j-1} = c_1(AB)_1 + \ldots + c_{j-1}(AB)_{j-1}$ which proves the claim.
(b) Take A_1 to be the identity matrix and A_2 to be the all zeros matrix.

Problem 10 *Monday 2/26*

Suppose the $m \times n$ matrix A ($m < n$) has a *right inverse* B, that is, a matrix B such that $AB = I$, the identity.

(a) What must the dimensions (the height and width) of B and of I be?
(b) Try calculating B in `MATLAB`: let $A = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 0 & 2 \end{bmatrix}$ and use `A\I`. (This is the code in `MATLAB` for finding a matrix B such that $AB = I$. The $k \times k$ identity matrix is `eye(k)` in `MATLAB`.)
(c) Now try calculating B another way, with `rref([A I])`. (This is the reduced row echelon form, the result of Gauss-Jordan elimination.) What do you get? Use your result to state another, different, B with $AB = I$. Why is B not unique?
(d) Why can't there be a left inverse $CA = I$?

Solution 10

(a) B is n-by-m, and I is m-by-m.
(b) I get $\begin{bmatrix} 0.2 & -0.1 \\ 0 & 0 \\ 0.2 & 0.4 \end{bmatrix}$ for B.
(c) I get $\begin{matrix} 1 & 0 & -1 & 0 & -0.5 \\ 0 & 1 & -1.6667 & -0.3333 & -0.6667 \end{matrix}$ for `rref([A I])`. The last two columns suggest to me that $B = \begin{bmatrix} 0 & -0.5 \\ -0.3333 & -0.6667 \\ 0 & 0 \end{bmatrix}$, and indeed $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
So B is not unique. Let b_1 and b_2 be the columns of B. Then looking for B is the same as solving $Ab_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Ab_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $m < n$ we know then that $r < n$ so there are free variables. This means that if there is a solution, this is not unique (because $N(A)$ is not just 0).
(d) A has rank m (or less), C has rank m (or less), so CA has rank at most m also. So it can't equal I, which has full rank n.
Moral: only square matrices can have both left and right inverses. (And if both inverses exist, they're the same!)