18.06 **FINAL EXAM** May 24, 2007

Your PRINTED name is: <u>SOLUTIONS</u>

Please circle your recitation:

(1)	M 2	2-131	A. Osorno
(2)	M 3	2-131	A. Osorno
(3)	M 3	2-132	A. Pissarra Pires
(4)	T 11	2-132	K. Meszaros
(5)	T 12	2-132	K. Meszaros
(6)	T 1	2-132	Jerin Gu
(7)	T 2	2-132	Jerin Gu

Grading

1	
2	
3	
4	
5	
6	
7	
8	
Total:	

Problem 1 *(10 points)*

Let $A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 6 & 6 & 3 & 3 \\ 3 & 4 & 2 & 2 \end{pmatrix}$.

(a) Calculate the dimensions of the 4 fundamental subspaces associated with A.

(b) Give a basis for each of the 4 fundamental subspaces.

(c) Find the complete solution of the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

Solution 1

(a) The rank is 2, so the dimensions are:

$C(A) \rightsquigarrow r = 2$

$C(A^T) \rightsquigarrow r = 2$

$N(A) \rightsquigarrow n - r = 4 - 2 = 2$

$N(A^T) \rightsquigarrow m - r = 3 - 2 = 1.$

(b) We can get these by elimination or by inspection:

$C(A) \rightsquigarrow \begin{pmatrix} 3 & 6 & 3 \end{pmatrix}^T, \begin{pmatrix} 2 & 6 & 4 \end{pmatrix}^T$

$C(A^T) \rightsquigarrow \begin{pmatrix} 3 & 2 & 1 & 1 \end{pmatrix}^T, \begin{pmatrix} 6 & 3 & 3 & 1 \end{pmatrix}^T$

$N(A) \rightsquigarrow \begin{pmatrix} 0 & -1/2 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -1/2 & 0 & 1 \end{pmatrix}^T$

$N(A^T) \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$

(c) $x = x_{particular} + x_{nullspace} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}.$

Problem 2 *(10 points)*

Consider the system of linear equations:

$$\begin{cases} x & + & y & + & z & = & 1 \\ 2x & & & + & z & = & 2 \\ -x & + & y & + & az & = & b \end{cases}$$

In parts (a)–(c) below circle correct answers. Explain your answers.

(a) For $a = 1$, $b = -1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(b) For $a = 0$, $b = 1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(c) For $a = 0$, $b = -1$, the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(d) Solve the system for $a = b = 1$.

Solution 2

If we eliminate the augmented matrix we get $\begin{pmatrix} 1 & 1 & 1 & & 1 \\ 0 & -2 & -1 & & 0 \\ 0 & 0 & a & & b+1 \end{pmatrix}$.

(a) **Exactly one solution:** The matrix is invertible.

(b) **No solutions:** Get a row of zeroes in the matrix with no zero in the augmented column.

(c) **Infinitely many solutions:** Get a row of zeroes with a zero in the augmented column.

(d) Using back substitution we get $x = \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^T$.

Problem 3 *(10 points)*

Let L be the line in $\mathbb{R}^3$ spanned by the vector $(1,1,1)^T$. Let P be the projection matrix for the projection onto the line L.

(a) What are the eigenvalues of the matrix P? (Indicate their multiplicities.)

(b) Find an *orthonormal* basis of the orthogonal complement $L^\perp$ to the line L.

(c) Calculate the projection of the vector $(1,2,3)^T$ onto the line L.

(d) Calculate the projection of the vector $(1,2,3)^T$ onto the orthogonal complement $L^\perp$.

Solution 3

(a) P is a projection matrix onto a subspace of dimension 1, so the eigenvalues are 1, 0, 0.

(b) $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$

(c) $p = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$

(d) The projection onto $L^\perp$ is $b - p = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

Problem 4 *(10 points)*

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

In parts (a)–(c) below circle correct answers. Explain your answers.

(a) The matrix A is singular: **True** **False**

(b) The matrix $A + 2I$ is singular: **True** **False**

(c) The matrix A is positive definite: **True** **False**

(d) Find all eigenvalues of A and the corresponding eigenvectors.

(e) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(f) Solve the system of differential equations $\frac{d\mathbf{u}(t)}{dt} = A\,\mathbf{u}(t)$, $\mathbf{u}(0) = (1, 0, 0)^T$.

Solution 4

(a) **True** ($\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T$ is in the nullspace).

(b) **True** ($\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$ is in the nullspace).

(c) **False** The matrix is singular so has 0 as eigenvalue.

(d) A is singular, so 0 is an eigenvalue with eigenvector $\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T$.

$A + 2I$ is singular, so -2 is an eigenvalue with eigenvector $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$.

We can get the last eigenvalue by looking a the trace: 6. The eigenvector is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$.

(e)

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{62} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{Q^T}.$$

(f) $u(t) = e^{At}u(0) = Qe^{\Lambda t}Q^T u(0) = \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3}e^{6t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 5 *(10 points)*

Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$.

(a) What is the rank of A?

(b) Calculate the matrix $A^T A$. Find all its eigenvalues (with multiplicities).

(c) Calculate the matrix AA^T. Find all its eigenvalues (with multiplicities).

(d) Find the matrix Σ in the singular value decomposition $A = U\Sigma V^T$.

Solution 5

(a) 1

(b) $A^T A = \begin{pmatrix} 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \end{pmatrix}$. It was rank 1 so the eigenvalues are 56, 0, 0, 0.

(c) $AA^T = \begin{pmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \\ 12 & 24 & 36 \end{pmatrix}$. It was rank 1 so the eigenvalues are 56, 0, 0.

(d) $\Sigma = \begin{pmatrix} \sqrt{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Problem 6 *(10 points)*

Let A_n be the tridiagonal $n \times n$-matrix with 2's on the main diagonal, 1's immediately above the main diagonal, 3's immediately below the main diagonal, and 0's everywhere else:

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \ddots & 0 \\ 0 & 3 & 2 & 1 & \ddots & 0 \\ 0 & 0 & 3 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix},$$

(a) Express the determinant $\det(A_n)$ in terms of $\det(A_{n-1})$ and $\det(A_{n-2})$.

(b) Explicitly calculate $\det(A_n)$, for $n = 1, \ldots, 6$.

Solution 6

(a) Using cofactors twice we get $\det(A_n) = 2\det(A_{n-1}) - 3\det(A_{n-2})$.

(b) $\det(A_1) = \det[2] = 2$.

$\det(A_2) = \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 1.$

$\det(A_3) = 2 \cdot 1 - 3 \cdot 2 = -4.$

$\det(A_4) = 2 \cdot (-4) - 3 \cdot 1 = -11.$

$\det(A_5) = 2 \cdot (-11) - 3 \cdot (-4) = -10.$

$\det(A_6) = 2 \cdot (-10) - 3 \cdot (-11) = 13.$

Problem 7 *(10 points)*

Calculate the determinant of the following 6×6-matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Solution 7

This determinant could be computed using cofactors or doing row operations to simplify and then cofactors. it could also be computed as follows.

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}}_{B}.$$

P is a permutation matrix with determinant -1.

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} - I.$$

The eigenvalues of the matrix with all 1's are 6, 0, 0, 0, 0, 0 so the eigenvalues of B are 5, -1, -1, -1, -1, -1, so the determinant of B is -5.

Thus the determinant of A is 5.

Problem 8 *(10 points)*

(a) Calculate A^{100} for $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(b) Calculate B^{100} for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(c) What will happen with the house (shown below) when we apply the linear transformation $T(v) = Bv$ for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Will it be dilated? Draw the picture of the transformed house.

Solution 8

(a) $A^2 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$ so $A^{100} = \begin{pmatrix} 5^{100} & 0 \\ 0 & 5^{100} \end{pmatrix}$.

(b) $B^4 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ so $B^{100} = \begin{pmatrix} (-4)^{25} & 0 \\ 0 & (-4)^{25} \end{pmatrix} = \begin{pmatrix} -4^{25} & 0 \\ 0 & -4^{25} \end{pmatrix}$.

(c) From part (b) we see that B^4 is stretching by 4 and rotating by π. Thus B is rotating by $\pi/4$ and stretching by $\sqrt{2}$.