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18.06 Linear Algebra

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Recitation 8
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Problem Set 1

Problem ★ Find the possible failures in the column picture and the row picture, and match them up. Success would be 3 columns whose combinations give every vector **b**, which matches with 3 planes in the row picture that intersect at one point (the unique solution **x**). Give numerical examples of these two types of failure:

- 3 columns lie on the same line; 3 planes are parallel
  (then if **b** happens to lie on that line of columns, the 3 planes meet in a ...)
- 3 columns in the same plane, but no two on the same line. Then 3 planes do what? Which **b**'s are okay?

Now give numerical examples of the other types of failure in the column and row pictures.

Solution. Let's consider all of the possible cases we can have in a 3×3 system.

1) **Success.** In order for a system to be successful through elimination, there may be two possibilities for the structure of the coefficient matrix.

   a) *Full Success.* The matrix may contain three independent columns. This way, the columns span all of space, and the planes formed by the rows intersect at a single point. An example of this is the following.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{★.1}$$

   b) *Temporary Failure.* The matrix may need a row exchange in order to allow for a pivot to be found. After this happens, three independent columns are present, and the columns span all of space. Therefore, the planes intersect at a single point. The following is an example.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{★.2}$$

2) **Failure**. There are two types total failure, each with two subtypes that describe the behavior of the column and row picture.

a) *Colinear Columns.* The coefficient matrix may have columns that are along the same line. This offers two possibilities.

i) The vector $\mathbf{b}$ may be on the line that the coefficient matrix columns are. If this is the case, there are infinitely many solutions because the columns have infinitely many combinations to produce the required vector. In the row picture, all three planes are superimposable. An example is the following.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix} \tag{$\bigstar$.3}$$

ii) The vector $\mathbf{b}$ may be elsewhere in space than along the particular line containing the matrix columns. If this is the case, there are no solutions to the system, and therefore no possible combination of the columns to produce the desired vector. In the row picture, the three planes are parallel. The following is an example.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 23 \end{bmatrix} \tag{$\bigstar$.4}$$

b) *Coplanar Columns.* The coefficient matrix may have columns that are in the same plane. This also offers two possibilities.

i) The vector $\mathbf{b}$ may be in the plane that the coefficient matrix columns are. If this is the case, there are infinitely many solutions because the columns have infinitely many combinations to produce the required vector. In the row picture, the three planes meet at a line. Below is an example of this.

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} \tag{$\bigstar$.5}$$

ii) The vector $\mathbf{b}$ may be outside of the plane that the coefficient matrix columns are. If this is the case, there is no combination of the column vectors that will produce the required vector, and thus there are no solutions to the system. In the row picture, the three planes meet in three different lines. Below is an example.

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \end{bmatrix} \tag{$\bigstar$.6}$$

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