

## 18.06 Solutions to Midterm Exam 3, Spring, 2001

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

(a) Given that one eigenvalue of  $A$  is  $\lambda = 6$ , find the remaining eigenvalues.

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$$\det \begin{pmatrix} 4-\lambda & -1 & 1 \\ -1 & 4-\lambda & -1 \\ 1 & -1 & 4-\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow -\lambda^3 + 12\lambda^2 - 45\lambda + 54 = 0$$

$$\Leftrightarrow \lambda_1 = 6, \lambda_2 = \lambda_3 = 3$$

(b) Find three linearly independent eigenvectors of  $A$ .

• For  $\lambda_1 = 6$ , we have  $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}$ , and so  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ .

For  $\lambda_2 = \lambda_3 = 3$ , we have  $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$ , and two linearly independent

solutions are  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  and  $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ .

(c) Find an *orthogonal* matrix  $Q$  and a diagonal matrix  $\Lambda$ , so that  $A = Q\Lambda Q^T$ .

• To obtain the first column of  $Q$ , let  $\mathbf{q}_1 = \mathbf{v}_1 / \|\mathbf{v}_1\| = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ . The second column

of  $Q$  is given by  $\mathbf{q}_2 = \mathbf{v}_2 / \|\mathbf{v}_2\| = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ . The vector  $\mathbf{v}_3$  is not orthogonal to  $\mathbf{q}_2$ , so

we need to use Gram-Schmidt to make it so:

$$\tilde{\mathbf{q}}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix}.$$

Normalising gives  $\mathbf{q}_3 = \tilde{\mathbf{q}}_3 / \|\tilde{\mathbf{q}}_3\| = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$ . Hence,

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions  $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$  and  $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$  of this system and hence

find the solution  $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$  which satisfies the initial condition  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

• We can write this system as  $\frac{d}{dt}\mathbf{x} = \begin{pmatrix} -7 & 2 \\ -6 & 0 \end{pmatrix}\mathbf{x}$ . The eigenvalues of this matrix are given by

$$\begin{aligned} \det \begin{pmatrix} -7 - \lambda & 2 \\ -6 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 + 7\lambda + 12 &= 0 \\ \Leftrightarrow \lambda_1 = -4, \lambda_2 = -3. \end{aligned}$$

The corresponding eigenvectors are: for  $\lambda_1 = -4$ ,  $\begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$ , so  $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ; and

for  $\lambda_2 = -3$ ,  $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}\mathbf{v}_2 = \mathbf{0}$ , so  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Hence the general solution to this system of equations is

$$\mathbf{x}(t) = C_1 e^{-4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To satisfy the initial condition, we need to solve

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives  $C_1 = 1$  and  $C_2 = -1$ .

3. (40pts.) Let  $A_n$  be the  $n \times n$  tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for  $n \geq 3$  that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \quad (1)$$

• Let us expand the determinant of  $A_n$  along the first column,

$$\begin{aligned} \det(A_n) &= 1 \cdot \det \begin{pmatrix} 1 & -a & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} + a \det \begin{pmatrix} -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det \begin{pmatrix} 1 & -a & 0 & \cdots \\ -a & 1 & -a & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det(A_{n-2}). \end{aligned}$$

Here, we have expanded the determinant in the second term of line 1 along the first row.

(b) Show that eq.(1) can equivalently be written as  $\mathbf{x}_n = B \mathbf{x}_{n-1}$ , where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

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$$\begin{aligned} \mathbf{x}_n &= \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} = \begin{pmatrix} \det(A_{n-1}) - a^2 \det(A_{n-2}) \\ \det(A_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \det(A_{n-1}) \\ \det(A_{n-2}) \end{pmatrix} = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \mathbf{x}_{n-1}. \end{aligned}$$

(c) For  $a^2 = \frac{3}{16}$ , find an expression for  $\det(A_n)$  for any  $n$ . (*Hint:* One method starts by writing  $B$  in the form  $B = SAS^{-1}$ , where  $\Lambda$  is a diagonal matrix.)

• The answer is given by  $\mathbf{x}_n = B^{n-2}\mathbf{x}_2$ . To determine  $\mathbf{x}_2$ , we need  $\det(A_1) = 1$ , and  $\det(A_2) = \det\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} = 1 - a^2 = 1 - \frac{3}{16} = \frac{13}{16}$ . To find  $B^{n-2}$ , we first need to diagonalise  $B$ . Eigenvalues are given by

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & -\frac{3}{16} \\ 1 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 - \lambda + \frac{3}{16} &= 0 \\ \Leftrightarrow \lambda_1 = \frac{3}{4}, \lambda_2 = \frac{1}{4}. \end{aligned}$$

The eigenvector corresponding to  $\lambda_1 = \frac{3}{4}$  is given by  $\begin{pmatrix} 1/4 & -3/16 \\ 1 & -3/4 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}$ , so that  $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . The eigenvector corresponding to  $\lambda_2 = \frac{1}{4}$  is given by  $\begin{pmatrix} 3/4 & -3/16 \\ 1 & -1/4 \end{pmatrix} \mathbf{v}_2 = \mathbf{0}$ , and hence  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ . So  $B = S\Lambda S^{-1}$ , where

$$S = \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad S^{-1} = \frac{1}{8} \begin{pmatrix} 4 & -1 \\ -4 & 3 \end{pmatrix}.$$

Hence,

$$\mathbf{x}_n = S\Lambda^{n-2}S^{-1}\mathbf{x}_2 = \frac{1}{8} \begin{pmatrix} \frac{27}{4} \left(\frac{3}{4}\right)^{n-2} - \frac{1}{4} \left(\frac{1}{4}\right)^{n-2} \\ 9 \left(\frac{3}{4}\right)^{n-2} - \left(\frac{1}{4}\right)^{n-2} \end{pmatrix}$$

and

$$\det(A_n) = \frac{2}{4^{n+1}} (3^{n+1} - 1).$$