18.06 Final Exam December 16, 1999 Closed Book

1 (12 pts) Let

$$A = \begin{bmatrix} 7 & 0 & 2 & 4 \\ 7 & 1 & 3 & 6 \\ 14 & -1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find bases for the four fundamental subspaces.

(b) Find the conditions on b_1, b_2, and b_3 so that

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution.

(c) If $Ax = b$ has a solution x_p, describe all of the solutions.

Solution: Write L and U for the two matrices on the right, so $A = LU$. L is invertible and U is the row reduced form of A.

(a) A basis for $C(A)$ is given by the pivot columns of A, $(7, 7, 14)$ and $(0, 1, -1)$. For $N(A)$: the special solutions, $(-2/7, -1, 1, 0)$ and $(-4/7, -2, 0, 1)$. For $C(A^{\mathrm{T}})$: the nonzero rows of U, $(7, 0, 2, 4)$ and $(0, 1, 1, 2)$. For $N(A^{\mathrm{T}})$, row reduce A^{T}. There is one special solution, $(-3, 1, 1)$.

(b) Row reduce the augmented matrix $[\,A\,|\,b\,]$. There is a solution when $b_3 + b_2 - 3b_1 = 0$.

(c) All solutions are of the form $x_p + c_1(-2/7, -1, 1, 0) + c_2(-4/7, -2, 0, 1)$, where c_1 and c_2 are real numbers.

2 (10 pts) Let A and B be any two matrices so that the product AB is defined.

(a) Explain why every column of AB is in the column space of A.

(b) How does part (a) lead to the conclusion that the rank of AB is less than or equal to the rank of A? State your reasoning in logical steps.

Solution:

(a) Write $a_1, \ldots, a_n$ for the columns of A. Then the first column of AB is $b_{11}a_1 + b_{21}a_2 + \cdots + b_{n1}a_n$. This is a linear combination of the columns of A, hence is in $C(A)$. The same reasoning shows that the other columns of AB are in $C(A)$.

(b) From part (a), $C(AB)$ is a subspace of $C(A)$. Therefore $\dim C(AB) \leq \dim C(A)$, or $\text{rank}(AB) \leq \text{rank}(A)$.

3 (10 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) What is the matrix A expressing T in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (The same basis is used for the input and the output.)

(c) What is the matrix B expressing T in terms of the basis consisting of eigenvectors of A? (The same basis is used for the input and output.) (There are two possible correct answers, depending on what order you pick the eigenvectors.)

Solution:

(a) $T\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = T\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right) - T\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -6 \\ 5 \end{smallmatrix}\right]$.

(b) $A = \left[\begin{smallmatrix} -4 & -6 \\ 3 & 5 \end{smallmatrix}\right]$.

(c) The eigenvalues of A are 2 and -1, so $B = \left[\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

4 (16 pts) Let V be the subspace of $\mathbf{R}^3$ consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$x + 2y - 5z = 0.$$

(a) Find a 3×2 matrix A whose column space is V.

(b) Find an orthonormal basis for V.

(c) Find the projection matrix P projecting onto the left nullspace (not the column space!) of A.

(d) Find the least squares solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: Parts (a), (b), and (d) have infinitely many possible correct answers.

(a) The columns of A should be two linearly independent solutions, to $x + 2y - 5z = 0$. For example, $A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Applying the Gram-Schmidt process to the columns of A yields $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(c) Let Q be the matrix whose columns are the vectors in part (b). The projection matrix onto the column space of A is QQ^{T}. Since the left nullspace is the orthogonal complement of the column space, its projection matrix is $I - QQ^{\mathrm{T}} = \frac{1}{30} \begin{bmatrix} 1 & 2 & -5 \\ 2 & 4 & -10 \\ -5 & -10 & 25 \end{bmatrix}$.

(d) The solution to $A^{\mathrm{T}} A \hat{x} = A^{\mathrm{T}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\hat{x} = \begin{bmatrix} -17/15 \\ 2/3 \end{bmatrix}$.

5 (15 pts) Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}$$

(a) Find all possible information about r, m, and n. (The rank and the shape of A.)

(b) Find an example of such a matrix A with r, m, and n all as small as possible.

(c) How do you know that $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is not in the nullspace of A^{T}?

Solution:

(a) $m = 3$, $0 < r < 3$, and $r < n$.

(b) With $r = 1$, $m = 3$ and $n = 2$: $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) The column space and the left nullspace are always orthogonal.

6 (13 pts) In each case give all the information you can about the eigenvalues and eigenvectors, when the matrix A has the following property:

(a) The powers A^k approach the zero matrix.

(b) The matrix is symmetric positive definite.

(c) The matrix is not diagonalizable.

(d) The matrix has the form $A = uv^{\mathrm{T}}$, where u and v are vectors in $\mathbf{R}^3$. (You might want to try an example.)

(e) A is similar to a diagonal matrix with diagonal entries 1, 1, and 2.

Solution:

(a) The eigenvalues are between -1 and 1.

(b) The eigenvalues are positive. There are n linearly independent, orthogonal eigenvectors.

(c) There is a repeated eigenvalue and that eigenvalue has fewer linearly independent eigenvectors than its multiplicity as a root of the characteristic polynomial.

(d) The eigenvalues are 0 (the eigenvectors are all vectors orthogonal to v) and $u \cdot v$ (the eigenvectors are the multiples of u).

(e) The eigenvalues are 1 and 2 and there are three linearly independent eigenvectors (two for $\lambda = 1$ and one for $\lambda = 2$).

7 (12 pts) Define a sequence of numbers in the following way: $G_0 = 0$, $G_1 = 1/2$, and $G_{k+2} = (G_{k+1} + G_k)/2$. (Each number is the average of the two previous numbers.)

(a) Set up a 2×2 matrix A to get from $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ to $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$.

(b) Find an explicit formula for G_k.

(c) What is the limit of G_k as $k \to \infty$?

Solution:

(a) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$.

(b) A has eigenvalues 1 and $-1/2$ with corresponding eigenvectors $(1,1)$ and $(1,-2)$. Write the initial state as a linear combination of the eigenvectors: $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Applying A^k, we get $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \frac{1}{3}(1)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}(-\frac{1}{2})^k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so $G_k = \frac{1}{3}(1 - (-\frac{1}{2})^k)$.

(c) As $k \to \infty$, $G_k \to \frac{1}{3}$.

8 (12 pts) (a) Suppose A is a 4×4 matrix of rank 3, and let

$$x = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \end{bmatrix}$$

be the cofactors of its first row. Explain why $Ax = 0$. (So the cofactors give a formula for a nullspace vector!)

Hint: The first component of Ax and the second component of Ax are determinants of (different) matrices. What are these matrices and why do they have zero determinants? (The 3rd and 4th components of Ax follow similarly, so you can just answer for the 1st and 2nd components.)

(b) Compute the determinant of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Hint: You might find it convenient to use the fact that the columns are orthogonal.

Solution:

(a) The first component of Ax is $\det A$, which is zero because A has rank 3. The second component of Ax is the determinant of the matrix obtained by replacing the first row of A with the second row. This matrix has repeated rows, so its determinant is zero.

(b) $\det B = 16$. You can compute this by brute force. Another way is to note that $B^2 = 4I$, so the eigenvalues of B are ± 2. Since the trace of B is zero, the eigenvalues are $\lambda = 2$, 2, -2, and -2, so $\det B = 16$.