**3 December 1997**                    **Profs. S. Lee and A. Kirillov**

# MASSACHUSETTS INSTITUTE OF TECHNOLOGY

## Hour Exam III for Course 18.06: <u>Linear Algebra</u>

Recitation Instructor:                    Your Name: **SOLUTIONS**


Recitation Time:                          Lecturer:


Grading

1. **30**

2. **22**

3. **16**

4. **32**

**TOTAL: 100**

**Show all your work on these pages.**
**No calculators or notes.**
**Please work carefully, and check your intermediate results.**
**Point values (total of 100) are marked on the left margin.**

**1.** Let $A = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$.

[10]   **1a.** Find the eigenvalues of $A$.

$$\boxed{\lambda_1 = 4 + i, \quad \lambda_2 = 4 - i.}$$

Find roots of quadratic equation:
$\det(A - \lambda I) = (4 - \lambda)(4 - \lambda) - (-1) = \lambda^2 - 8\lambda + 17 = 0.$

[10]   **1b.** Find an eigenvector for each eigenvalue of $A$.

$$\boxed{x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.}$$

$$\underbrace{\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}}_{(A-\lambda_1 I)} \underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \underbrace{\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}}_{(A-\lambda_2 I)} \underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[10]    **1c.** Compute $x_1^H x_2$.
(Note: $x_1$ and $x_2$ are the complex eigenvectors that you obtained in **1b**.)

$$\boxed{x_1^H x_2 = 0.}$$

**2.** Let $A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$.

[12]   **2a.** Find an invertible matrix $S$ that makes $S^{-1}AS$ a diagonal matrix.

$$S = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

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The diagonal entries of A are its eigenvalues.
The columns of S are the eigenvectors of A for λ₁ = −2, λ₂ = 0.
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[10]  **2b.**  For the differential equation $\frac{du}{dt} = Au$, give a nonzero initial vector $u(0)$ such that $u(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ or any nonzero multiple of it.}$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Choose $c_2 = 0$ and $c_1 \neq 0$ for initial vector to approach $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

[16]   **3.** Fill in the matrix $A = \begin{bmatrix} 0.5 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ so that $A$ is a positive Markov matrix with the

steady state vector $x_1 = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$.

(Recall that the limit of $A^k u_0$ is always a multiple of $x_1$.)

$$A = \begin{bmatrix} 1/2 & 1/6 \\ 1/2 & 5/6 \end{bmatrix}.$$

The first column of A adds to 1 when $a_{21} = 1/2$.
Next, we solve $A = \begin{bmatrix} 1/2 & a_{12} \\ 1/2 & a_{22} \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$ for the second column of $A$.

Along the first row:  $(1/8) + (3/4)a_{12} = 1/4$, which gives $a_{12} = 1/6$.
The second column of A adds to 1 for $a_{22} = 5/6$.
Observe that the steady state vector satisfies $Ax_1 = (1)x_1$.

**4.** Each <u>independent</u> question refers to the matrix $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

In each case, find the value of $d$ that makes the statement true (and show your work!).

[10]   **4a.** Give a value for $d$ such that $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector of $A$.

$\boxed{d = 41/25\,.}$

$$\begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 5d - 4 \end{bmatrix} = (21/5)\begin{bmatrix} 5 \\ 1 \end{bmatrix}\,.$$

`In the second component, solve` $5d - 4 = (21/5)$ `for` $d$.

[10]   **4b.** Give a value for $d$ such that 2 is one of the eigenvalues of $A$.

$\boxed{d = -12\,.}$

`When 2 is an eigenvalue of` $A$,

$$A - 2I = \begin{bmatrix} 2 & 1 \\ d & -6 \end{bmatrix} \quad \text{must have linearly dependent columns.}$$

[12]    **4c.** Give a value for $d$ such that $A$ is a nondiagonalizable matrix.

Recall that $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

$\boxed{d = -16.}$

The issue of nondiagonalizability only comes up for a matrix that has some
repeated eigenvalues.
In this case, 0 is a twice repeated eigenvalue of $A$ when $d = -16$.
The eigenvalue is repeated twice, but we only find one linearly independent

eigenvector (via the special solution to $(A - 0I)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).