MIT 18.06 Exam 3, Fall 2022
Johnson

Your name: ______________________________
(*printed*)

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Recitation: ______________________________

Problem 1 [10+(4+4)+5+10 points]:

Two of the eigenvectors of the **real** matrix A are $x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2 + i$.

(a) Another eigenvalue of A is $\lambda_3 =$ ____, and A is a _ $\times$ _ matrix equal to $A =$ ____. You can leave your answer for A as a product of matrices and/or matrix inverses without simplifying.

(b) $\det A =$ ____ and $\operatorname{trace} A =$ ____.

(c) $\det(A - \lambda I) =$ ____ (simplify to a polynomial in λ). (Time-saving hint: You can do this without calculating A explicitly!)

(d) Give *all* of the eigenvalues, and corresponding eigenvectors, of $(A^2 - 2I)e^{(A^{-1})}$. You can leave your eigenvalues as **non-simplified** arithmetic expressions.

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Problem 2 [11+11+11 points]:

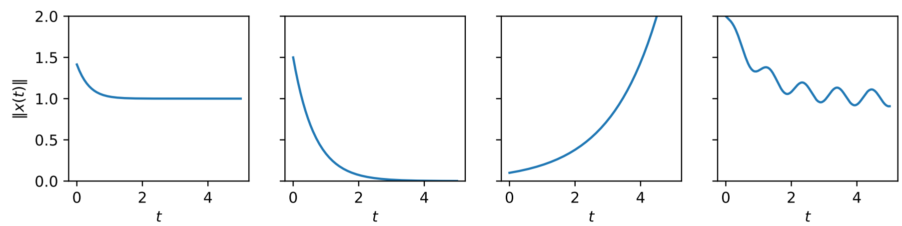
Consider the differential equation

$$\frac{dx}{dt} = -B^T B x, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 3 \\ 4 & 0 & 4 \\ 5 & 1 & 5 \end{pmatrix}.$$

(a) $x(t) =$ (constant vector) is a possible solution of this ODE for what vector(s) x? (Describe *all* possible answers. Look carefully at B!)

(b) Which of the following looks like a possible plot of $\|x(t)\|$ versus t for some initial $x(0)$? Circle **all possibilities**. (Note: all vertical axes are identical.)

You know this because the eigenvalues of ____ must be ____.



(c) For $x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, give a **good approximation** for $x(1000) \approx$ ____.

(Give a specific numerical vector, no unknowns.)

(blank page for your work if you need it)

Problem 3 [10+8+8+8 points]:

Suppose that the sequence of vectors $y_0, y_1, y_2, \ldots \in \mathbb{R}^m$ satisfies the recurrence

$$\frac{y_n - y_{n-1}}{h} = A\left(\frac{y_{n-1} + y_n}{2}\right)$$

for some real $h > 0$ and some $m \times m$ matrix A.

(a) Write $y_n = (\_\_\_\_)y_{n-1} = (\_\_\_\_)y_0$, where you fill in the blanks with **some matrices** written in terms of A, I (the $m \times m$ identity), h, and n.

(b) If $y_0 = x_k$ where x_k is an **eigenvector** of A with eigenvalue λ_k, give a *much simpler* formula $y_n = \_\_\_\_$ in terms of x_k, λ_k, h, n.

(c) The solutions y_n **must** be **decaying to zero** as $n \to \infty$ if A is (**circle all** that apply): *real, Hermitian, positive-definite, positive-semidefinite, negative-definite, negative-semidefinite.* **Justify** your answer (briefly!).

(d) If $A = iB$ where B is **Hermitian** and **invertible**, then the solutions y_n for $y_0 \neq 0$ must be (**circle one**): growing, decaying to zero, approaching a nonzero constant, oscillating. **Justify** your answer (briefly!).

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