MIT 18.06 Exam 1, Fall 2022
Johnson

Your name: ______________________________
(*printed*)

Student ID: ______________________________

Recitation: ______________________________

Problem 1 (6+6+6+6+6+6=36 points):

Fill in the blanks:

(a) *Any* solution x to $Ax = b$ (if it exists) is always a sum of a vector in the ______ space of A plus a vector in the ______ space of A.

(b) $Ax = b$ is solvable if (and only if) b is orthogonal to every vector in the ______ space of A.

(c) If A is a 4×3 matrix and $Ax = b$ is *not* solvable for some b and the solutions are *not* unique when they exist, possible values for the rank of A are ______ (list all possibilities).

(d) $C(AB)$ *must* ______ (**contain** $\supseteq$ / **be contained in** $\subseteq$ / **equal** $=$) the column space of ______ (A **or** B) for *all* 4×4 matrices A and B.

(e) If $x, y, z \in \mathbb{R}^n$ are n-component vectors, then the number of operations to compute xy^Tz scales proportional to ______ (n, n^2, or n^3) for large n if you compute it in the order $(xy^T)z$, or proportional to ______ (n, n^2, or n^3) if you compute it in the order $x(y^Tz)$.

(f) If x_1 and x_2 are *both* solutions to $Ax = b$, then the vector $x_1 - x_2$ must be in the ______ space of A.

(blank page for your work if you need it)

Problem 2 (6+11+6+11=34 points):

If

$$\underbrace{\begin{pmatrix} 1 & 2 & 4 & 2 & 5 \\ & 2 & 3 & 5 & 6 \\ & & 3 & 4 & 3 \\ & & & 4 & 3 \\ & & & & 5 \end{pmatrix}}_{A} B \underbrace{\begin{pmatrix} 4 & 1 & 1 \\ & 1 & 1 \\ & & 2 \end{pmatrix}}_{C} x = b$$

has the complete solution

$$x = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \alpha_1 \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix},$$

for any scalar α_1, then:

(a) What is the size and rank of B?

(b) The ______ space of B must be spanned by the basis ______.

(c) In part (b), you could **alternatively** have found a basis for the ______ space of B, which is also fully determined by the information given because it is ______ to your answer from (b).

(d) Give a possible matrix B.

(blank page for your work if you need it)

Problem 3 (5+6+13+6=30 points):

Consider the matrix $A = BCD$ given by:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 2 & 0 \\ & 1 & 0 & 3 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & -2 & \\ & & & 3 \end{pmatrix}}_{D}.$$

(a) Write A^{-1} in terms of B^{-1}, C^{-1}, and D^{-1} (without computing any numbers).

(b) To compute the **sum** x of the four columns of A^{-1}, you could solve $Ax = b$ for x using what right-hand-side vector b?

(c) Compute the sum of the columns of A^{-1}.

(d) A basis for the column space $C(A)$ is ______.

(blank page for your work if you need it)