Problem Set 5 Solutions

Problem 1: Consider four non-zero vectors $\boldsymbol{c}$, $\boldsymbol{n}$, $\boldsymbol{r}$, $\boldsymbol{l}$ in $\mathbb{R}^2$. What are the conditions these four vectors need to satisfy, in order for:

- $\boldsymbol{c}$ to span the column space of A
- $\boldsymbol{n}$ to span the nullspace space of A
- $\boldsymbol{r}$ to span the row space of A
- $\boldsymbol{l}$ to span the left nullspace of A

for some 2×2 matrix A. If these conditions are satisfied, write down such a matrix A. (keep the given vectors $\boldsymbol{c}$, $\boldsymbol{n}$, $\boldsymbol{r}$, $\boldsymbol{l}$ abstract, i.e. don't just plug in numbers). *(20 points)*

Solution: By the orthogonality relations, we must have vanishing dot products

$$\boldsymbol{c} \cdot \boldsymbol{l} = 0.$$

$$\boldsymbol{n} \cdot \boldsymbol{r} = 0.$$

Suppose that both of these conditions are satisfied, and denote the entries of $\boldsymbol{r}$ and $\boldsymbol{c}$ by the formulae

$$\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \text{ and}$$

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Consider then the matrix

$$A = \boldsymbol{c}\boldsymbol{r}^T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} r_1 c_1 & r_2 c_1 \\ r_1 c_2 & r_2 c_2 \end{bmatrix}.$$

It is clear that the row space of A is in the span of $\boldsymbol{r}$, and that the column space of A is in the span of $\boldsymbol{c}$. By the assumption that $\boldsymbol{c}$ is not zero, the row space of A is equal to the span of $\boldsymbol{r}$, and similarly the assumption that $\boldsymbol{r}$ is not zero implies that the column space of A is equal to the span of $\boldsymbol{c}$.

By the results proved in class, the left null space of A consists of all vectors perpendicular to $\boldsymbol{c}$. This is a 1-dimensional space, so it is spanned by any non-zero element within it, such as $\boldsymbol{l}$. Similarly, the nullspace of A will be the 1-dimensional subspace of vectors perpendicular to $\boldsymbol{r}$. It follows that it is spanned by any non-zero vector perpendicular to $\boldsymbol{r}$, such as $\boldsymbol{n}$.

Grading rubric: 7 points for identifying the two dot products that must be 0. 7 points for writing down the correct matrix A. 6 points for correct explanations of why A satisfies the 4 required conditions.

Problem 2: Consider the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} -5 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Invent an algorithm (explain all the steps of the algorithm in words, and explain why it works) which takes general vectors $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ as inputs, and decides whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$. *(15 points)*

Solution: To decide whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$, you need to check two things:

- $\boldsymbol{p}$ lies in the plane spanned by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, i.e. there exist numbers α, β, γ such that:

$$\alpha \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ 3 \\ 2 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{1}$$

  Equating coefficients implies $-\alpha = p_4$, which forces $\alpha = -p_4$, and then:

$$-4\alpha + 3\beta = p_2 \quad \text{and} \quad 3\alpha + 2\beta = p_3$$

  i.e.:

$$\beta = \frac{p_2 - 4p_4}{3} \quad \text{and} \quad \beta = \frac{p_3 + 3p_4}{2}$$

  This is only possible if $\boxed{\frac{p_2 - 4p_4}{3} = \frac{p_3 + 3p_4}{2}}$. There is no condition on the first entry of equality (1), since you can always pick γ to make the equality hold.

- $\boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all three of the given vectors. This means one needs to check the conditions:

$$\begin{aligned} &2(b_1 - p_1) - 4(b_2 - p_2) + 3(b_3 - p_3) - (b_4 - p_4) = 0 \\ &-5(b_1 - p_1) + 3(b_2 - p_2) + 2(b_3 - p_3) = 0 \\ &-2(b_1 - p_1) = 0 \end{aligned}$$

Grading rubric: Any correct solution should obtain maximum points. Please be generous with partial credit for solutions that go halfway, but could be made to work.

Problem 3: Consider the line L spanned by $\begin{bmatrix}1\\3\\6\end{bmatrix}$ and the plane $V = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ such that } x + 3y + 6z = 0 \right\}$.

1. Compute any two basis vectors of the plane V. *(5 points)*

2. Compute the projection matrices P_L onto L and P_V onto V. *(10 points)*

3. Compute $P_L + P_V$. The answer should be a very nice matrix. Explain geometrically why you get this answer (hint: it has to do with the relationship between L and V). *(10 points)*

Solution:

1. We may use, for example, $\begin{bmatrix}6\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\2\\-1\end{bmatrix}$. Both of these vectors satisfy the condition $x+3y+6z = 0$, and so are elements of V. Since these vectors are not multiples of one another, they are linearly independent, and so span all of the 2-dimensional plane V. Many bases work equally well. Since V is the nullspace of $\begin{bmatrix}1 & 3 & 6\end{bmatrix}$, we could for example compute a basis by using the general algorithm from class for bases of nullspaces.

Grading rubric: 5 points for a correct basis with some justification. 3 points if the basis is correct but there is no justification.

2. Let A denote the matrix $\begin{bmatrix}6 & 0\\0 & 2\\-1 & -1\end{bmatrix}$. Since V is the column space of A, and the columns of A are linearly independent, we may use the formula from class to compute

$$P_V = A(A^TA)^{-1}A^T = \frac{1}{46}\begin{bmatrix}45 & -3 & -6\\-3 & 37 & -18\\-6 & -18 & 10\end{bmatrix}.$$

Similarly, let B denote the matrix $\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$, so that L is the column space of B. We calculate.

$$P_L = B(B^TB)^{-1}B^T = B(46)^{-1}B^T = \frac{1}{46}\begin{bmatrix} 1 & 3 & 6 \\ 3 & 9 & 18 \\ 6 & 18 & 36 \end{bmatrix}.$$

Grading rubric: 4 points for calculating P_L correctly. 4 points for calculating P_V correctly using whatever basis was determined in part 1 (even if this basis is incorrect because of a mistake in part 1). 2 points for justifying why the calculation of P_V works.

3. We see that $P_L + P_V$ is the 3×3 identity matrix. This is because the vector L is normal (perpendicular) to the plane V. Projecting a vector $\mathbf{w}$ onto the plane V is accomplished by subtracting a vector orthogonal to the V, specifically $P_L\mathbf{w}$, from $\mathbf{w}$.

Grading rubric: 5 points for writing that L is perpendicular for V. The remaining 5 points for noticing that $P_L + P_V$ is the identity. They should notice that this is supposed to be true even if they do earlier parts wrong, so no credit for simply adding incorrect P_L and P_V from an incorrect part 2.

Problem 4: Consider the following lines L_1 and L_2 in 3-dimensional space:

$$L_1 = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} \text{ for } x \in \mathbb{R} \right\} \qquad \text{and} \qquad L_2 = \left\{ \begin{bmatrix} y \\ 2y-1 \\ 3y \end{bmatrix} \text{ for } y \in \mathbb{R} \right\}$$

1. Which of these is a subspace and which is not? *(5 points)*

2. Use least squares to compute the smallest possible distance from a point on the line which is not a subspace to the line which is a subspace. *(5 points)*

3. By minimizing the quantity in part (2), find the points $P \in L_1$ and $Q \in L_2$ for which the distance $|PQ|$ is minimal among all possible choices of a point on either line. *(5 points)*

4. What can you say about the line PQ in relation to the lines L_1 and L_2? *(5 points)*

Solution:

1. L_1 is a subspace since it is the column space of the matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. On the other hand, L_2 is not a subspace because it does not contain the origin.

Grading rubric: 2 points for a correct justification that L_1 is a subspace. 3 points for a correct justification that L_2 is not a subspace.

2. Fix a number y, and consider the fixed point $Q = \begin{bmatrix} y \\ 2y-1 \\ 3y \end{bmatrix}$ in L_2. We seek to determine the distance of Q from L_1.

Let P denote the point on L_1 that is closest to Q. This is just the projection of Q onto L_1. Let a denote

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

so that a is a basis of L_1. Using the formula from class, we calculate

$$P = P_{L_1}Q = a\frac{a^TQ}{a^Ta} = a\frac{y+2y-1+3y}{3} = a\frac{6y-1}{3} = \begin{bmatrix} 2y-1/3 \\ 2y-1/3 \\ 2y-1/3 \end{bmatrix}$$

The distance between P and Q is then given by

$$|PQ| = \sqrt{(2y-1/3-y)^2 + (2y-1/3-(2y-1))^2 + (2y-1/3-3y)^2} = \sqrt{(y-1/3)^2 + (2/3)^2 + (-y-1/3)^2}$$

This simplifies to

$$|PQ| = \sqrt{y^2 - 2y/3 + 1/9 + 4/9 + y^2 + 2y/3 + 1/9} = \sqrt{2y^2 + 6/9} = \sqrt{2y^2 + 2/3}.$$

Grading rubric: 2 points for correct answer. 3 points for a correct explanation.

3. The quantity $\sqrt{2y^2+2}$ is minimized when $y = 0$, which is when Q is the point

$$Q = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

In the previous problem we also calculated the projection of Q onto L_1 to be given by

$$P = \begin{bmatrix} 2y - 1/3 \\ 2y - 1/3 \\ 2y - 1/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \end{bmatrix}.$$

Grading rubric: 3 points for correctly finding the value of Q that minimizes what was calculated in 2. 2 points for finding the corresponding point P.

4. The line PQ is perpendicular to both L_1 and L_2. For example, if the angle with L_1 were not $\frac{\pi}{2}$, it would be possible to move P while fixing Q and obtain a strictly shorter distance $|PQ|$.

Grading rubric: 5 points for noting that PQ is perpendicular to L_1 and L_2. No justification needed.

Problem 5: The equation of a parabola in the plane is $y = ax^2 + bx + c$.

1. Compute a, b, c such that the parabola passes through the points $(1, 0)$, $(2, 4)$, $(-1, -2)$ (don't just guess, use linear algebra to solve for a, b, c). *(10 points)*

2. Compute a, b, c such that the parabola is the best fit for the points $(1, 0)$, $(2, 4)$, $(-1, -2)$, $(-2, 5)$: this means that the sum of the squares of the vertical distances between the parabola and the four given points should be minimum (Hint: this is done similarly to the example of fitting a line, that we did at the end of Lecture 13). *(10 points)*

Solution:

1. Plugging in the point $(1, 0)$, we learn that

$$0 = a + b + c.$$

Plugging in the point $(2, 4)$, we learn that

$$4 = 4a + 2b + c.$$

Finally, plugging in the point $(-1, -2)$, we learn that

$$-2 = a - b + c$$

We can solve this by row reducing the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 4 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 4 \\ 0 & -2 & 0 & -2 \end{bmatrix}$$

From the last row we see that $-2b = -2$, so $b = 1$. From the middle row we see that

$$4 = -2b - 3c = -2 - 3c,$$

and so $c = 2$. Finally, the top row says that $0 = a + b + c = a - 1 + 2$, so $a = -1$.

The final answer is $a = -1, b = 1$, and $c = 2$.

Grading rubric: 5 points for setting up the correct system of equations. 3 points for attempting to solve the system using some techniques from this class. 2 points for the correct final answer.

2. Plugging in the first three points, we obtain as in part 1 the equations

$$a + b + c = 0$$

$$4a + 2b + c = 4$$

$$a - b + c = -2$$

Plugging in the additional point $(-2, 5)$ gives the equation

$$4a - 2b + c = 5.$$

Let A denote the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 1 \end{bmatrix},$$

and let b denote the vector

$$b = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 5 \end{bmatrix}.$$

We would like to solve the equation

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = b,$$

but no solutions exist. We can find the best fit parabola, according to least squares distance, by solving instead the equation

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = P_{C(A)} b.$$

By the formula $P_{C(A)} = A(A^TA)^{-1}A^T$, the desired vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can be calculated as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^TA)^{-1}A^Tb = \frac{1}{6}\begin{bmatrix} 11 \\ 0 \\ -17 \end{bmatrix}.$$

Thus, the best fit parabola, according to sum of squares of vertical distance, is given by

$$y = \frac{11x^2 - 17}{6}$$

Grading rubric: 5 points for setting up the correct 4 linear equations. 3 points for realizing the formula $(A^TA)^{-1}A^Tb$ with the correct values of the matrix A and vector b. 2 points for calculating correctly at the end.