Recitation 3. September 24

Focus: vector spaces and subspaces, the column space and null space of a matrix

A ***vector space*** is a set V in which you may add and scale vectors; a ***subspace*** of V is a subset of V which is closed under addition of vectors and scalar multiplication.

Let A be an $m \times n$ matrix. The ***column space*** $C(A)$ of A is the span of its columns; it is a subspace of $\mathbb{R}^m$. The ***null space*** $N(A)$ of A the set of vectors $\boldsymbol{v}$ such that $A\boldsymbol{v} = \mathbf{0}$; it is a subspace of $\mathbb{R}^n$.

1. Determine whether the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$:

   (a) The set of vectors of the form $\begin{bmatrix} 1 \\ -1 \\ a \end{bmatrix}$, where a is some real number.

   (b) The set $\{\mathbf{0}\}$ consisting of only the zero vector.

   (c) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the equation $4x - 3y + 2z = 0$.

Solution: (a) This set is not a subspace. For instance, $\begin{bmatrix} 1 \\ -1 \\ a \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ a+b \end{bmatrix}$ for any $a, b \in \mathbb{R}$, which is not in the set. (Also, this set does not contain $\mathbf{0}$.)

(b) This set is a subspace. It is closed under addition because $\mathbf{0}+\mathbf{0} = \mathbf{0}$, and closed under scalar multiplication because $c\mathbf{0} = \mathbf{0}$ for any $c \in \mathbb{R}$.

(c) This set is a subspace. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ are elements of the set, then $4x-3y+2z = 0$ and $4x'-3y'+2z' = 0$. However, then

$$(4x - 3y + 2z) + (4x' - 3y' + 2z') = 0 \Leftrightarrow 4(x + x') - 3(y + y') + 2(z + z') = 0,$$

so $\begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ is in the set. If $c \in \mathbb{R}$, then

$$c(4x - 3y + 2z) = 0 \Leftrightarrow 4(cx) - 3(cy) + 2(cz) = 0,$$

so $\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} = c\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the set. (Note also that this set is the null space of $\begin{bmatrix} 4 & -3 & 2 \end{bmatrix}$.)

2. Let

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -7 & -1 \\ -3 & 12 & 3 \end{bmatrix}.$$

   Determine the space W of vectors $\boldsymbol{b}$ such that $A\boldsymbol{v} = \boldsymbol{b}$ has a solution. Find a lower triangular 3×3 matrix whose column space is W.

Solution: This space W is precisely the column space of A, i.e. the linear span of $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -7 \\ 12 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$.

However, $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ -7 \\ 12 \end{bmatrix}$ (doing column operations, for instance, would show you this), so W equals the linear span of just $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -7 \\ 12 \end{bmatrix}$. (That is, vectors of the form $\begin{bmatrix} a-3b \\ 2a-7b \\ -3a+12b \end{bmatrix}$ for some real numbers a, b.) We know from the above discussion that $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ span W, so the lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 3 & 0 \end{bmatrix}$$

has column space W.

3. Use Gauss-Jordan elimination to compute the null space $N(X)$ of the matrix

$$X = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 0 & -2 & 1 \end{bmatrix}.$$

Solution: We perform Gauss-Jordan elimination on X:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -8 & 3 & 5 \\ 0 & 4 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -8 & 3 & 5 \\ 0 & 0 & -\frac{5}{2} & \frac{7}{2} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{3}{8} & -\frac{5}{8} \\ 0 & 0 & 1 & -\frac{7}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\frac{7}{5} \\ 0 & 1 & 0 & -\frac{23}{20} \\ 0 & 0 & 1 & -\frac{7}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{9}{10} \\ 0 & 1 & 0 & -\frac{23}{20} \\ 0 & 0 & 1 & -\frac{7}{5} \end{bmatrix},$$

so the null space consists of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that

$$a = -\frac{9}{10}d, \qquad b = \frac{23}{20}d, \text{ and } c = \frac{7}{5}d.$$

It is therefore the space of vectors $\alpha \begin{bmatrix} -18 \\ 23 \\ 28 \\ 20 \end{bmatrix}$, where $\alpha \in \mathbb{R}$.

4. Let B be an $m \times n$ matrix. Show that if $\boldsymbol{v} \in C(B^T)$, then $\boldsymbol{v} \cdot \boldsymbol{u} = 0$ for any $\boldsymbol{u} \in N(B)$.

Solution: Let $\boldsymbol{u} \in N(B)$. If $\boldsymbol{v} \in C(B^T)$, then $\boldsymbol{v} = B^T\boldsymbol{w}$ for some $\boldsymbol{w} \in \mathbb{R}^m$. Then,

$$\boldsymbol{v} \cdot \boldsymbol{u} = \boldsymbol{v}^T\boldsymbol{u} = (B^T\boldsymbol{w})^T\boldsymbol{u} = \boldsymbol{w}^T B\boldsymbol{u}.$$

However, because $\boldsymbol{u} \in N(B)$, $B\boldsymbol{u} = 0$, so the above expression equals 0.