MIT 18.06 Final Exam Solutions, Fall 2017
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Problem 1 (15 points):

A matrix $A = LU$ has the LU factors

$$L = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 0 & -2 & 1 & \\ -1 & -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -2 & 0 \\ & 1 & 0 & -2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

(a) If $b = \begin{pmatrix} -1 \\ 2 \\ 2 \\ -4 \end{pmatrix}$, what is $x = A^{-1}b$?

(b) Assuming you solved the previous part efficiently, roughly how much more arithmetic operations would be required for the same approach if the matrices were 8×8 instead of 4×4? It should be about ______ times more.

(c) If you form a new 4×5 matrix $B = \begin{pmatrix} A & b \end{pmatrix}$ by appending the vector b (from above) as an extra column after A, and perform the *same* elimination steps as were used to get the LU factors above, what upper-triangular matrix would you obtain? (Hint: if you did part (a) properly, this part can be done with *no arithmetic*.)

Solution:

(a) If $x = A^{-1}b$, then $b = Ax = LUx$. So, to solve for x we can solve the equation $b = Ly$ for y and then $y = Ux$ for x, and these can be solved **by forward/back substitution**. The forward substitution procedure for solving for y gives:

$$y_1 = -1$$

$$-2y_1 + y_2 = 2 \implies y_2 = 2 + 2y_1 = 2 + 2 * (-1) = 0$$

$$0y_1 - 2y_2 + y_3 = 2 \implies y_3 = 2 + 2y_2 = 2 + 2 * 0 = 2$$

$$-y_1 - y_2 - 2y_2 + y_4 = -4 \implies y_4 = -4 + y_1 + y_2 + 2y_3 = -4 + (-1) + 0 + 2*2 = -1$$

Then we can solve for x using back substitution:

$$x_4 = -1$$

$$x_3 = 2$$

$$x_2 - 2x_4 = 0 \implies x_2 = 2x_4 = -2$$

$$x_1 - x_2 - 2x_3 = -1 \implies x_1 = -1 + x_2 + 2x_3 = -1 + (-2) + 2 * 2 = 1$$

So the solution is $\boxed{x = (1, -2, 2, -1)}$.

Note: Contrary to popular opinion, you *cannot invert* a lower/upper-triangular matrix simply by flipping the signs below/above the diagonal. (This only works for very special triangular matrices, e.g. the "elementary" matrices representing a single elimination step.

You could also do this problem by multiplying LU to get A and then proceed with Gaussian elimination as usual. But this is just going in a circle: elimination on A will give you back U! Quantitatively, using this inefficient approach requires $\sim m^3$ operations rather than the $\sim m^2$ for forward/back-substitution.

(b) Back or forward substitution for $m \times m$ systems require m^2 operations. So, the ratio of the time required for $m = 8$ and $m = 4$ is about $8^2/4^2 = 2^2 = \boxed{4}$.

(c) Recall that row operations are given by multiplication on the left by an invertible matrix. The Gaussian elimination giving U from A is given by multiplication by an elimination matrix E, i.e. $EA = U$, and therefore $E = L^{-1}$. So, performing the same elimination steps to $B = \begin{pmatrix} A & b \end{pmatrix}$ gives $L^{-1}B = (L^{-1}AL^{-1}b) = (U, L^{-1}b)$. In part (1) we solved $Ly = b$ for y, so that $L^{-1}b = y = (-1, 0, 2, -1)$. So we obtain the matrix $\boxed{\begin{pmatrix} U & y \end{pmatrix}}$ with U as in the problem statement and y as computed in the solution to part (1).

Problem 2 (15 points):

You are given the recurrence relation

$$(2I + B^T B)x_{n+1} = (2I - B^T B)x_n$$

where B is a real 5×3 matrix. We start with a vector x_0 and compute $x_1, x_2, \ldots$.

(a) $x_n = A^n x_0$ for some matrix A (independent of x_0). What is A?

(b) If λ is an eigenvalue of $B^T B$, give an eigenvalue of A.

(c) **Circle all possible** behaviors of x_n for large n, given the information above: **decaying** to zero, **oscillating** but not growing or decaying in length, going to a **nonzero constant** vector, or **growing** longer and longer. Explain your answers by giving some property (or properties) that must be true of the eigenvalues of A.

(d) If $x_1 = 0$ for a nonzero x_0, give one of the singular values (σ) of B.

Solution:

(a) The matrix $B^T B$ is symmetric positive semidefinite by construction, so all of its eigenvalues are real and nonnegative. So the eigenvalues of $2I+B^T B$ are all real and at least 2, so in particular it has no eigenvalue equal to zero and hence is invertible. Therefore we can rearrange the recurrence relation to:

$$x_{n+1} = (2I + B^T B)^{-1}(2I - B^T B)x_n$$

and therefore $x_{n+1} = Ax_n$ for all n, where A is the marix $\boxed{A = (2I + B^T B)^{-1}(2I - B^T B)}$. To see that this really gives the right matrix, notice that we have $x_1 = Ax_0$, but also $x_2 = Ax_1 = AAx_0 = A^2 x_0$, $x_3 = Ax_2 = AA^2 x_0 = A^3 x_0$, etc.

It may be tempting to write "$\frac{2I+B^T B}{2I-B^T B}$" for A, but we don't use the expression "$\frac{C}{D}$" is not defined for matrices because it is ambiguous: is it $D^{-1}C$ or CD^{-1}? (Although in this particular case the numerator and denominator matrices commute.)

(b) Suppose v is an eigenvector for $B^T B$ with eigenvalue λ. Then v is also an eigenvector of $2I + B^T B$ and $2I - B^T B$, with eigenvalue $2 + \lambda$ and $2 - \lambda$, respectively, and so v is also an eigenvector of $(2I + B^T B)^{-1}$ with eigenvalue $1/(2 + \lambda)$. It follows that v is also an eigenvector for $A = (2I + B^T B)^{-1}(2I - B^T B)$ with eigenvalue $\boxed{\frac{2-\lambda}{2+\lambda}}$.

(c) As $B^T B$ is symmetric positive semidefinite all of its eigenvalues are real and nonnegative. If $\lambda \geq 0$ is a nonnegative number, then $-1 < (2 - \lambda)/(2 + \lambda) \leq 1$ (and you can't do any better than those upper and lower bounds). Note, in particular, that $\lambda = 0$ would give an eigenvalue of 1 for A, so this is a possibility. Furthermore, the argument in (2) actually implies that any basis of eigenvectors for $B^T B$ (which is diagonalizable because it is real symmetric) is also a basis of eigenvectors for A (with the eigenvalues transformed according to $\lambda \mapsto (2-\lambda)/(2+\lambda)$). All of the eigenvalues are real numbers in the half-open interval (-1,1], so the sequence $\{x_n\}_n$ cannot grow large or oscillate. It can decay, if all of the eigenvectors appearing in the decomposition of x_0 into a linear combination of eigenvectors have eigenvalue wth magnitude less than 1, and it can go to a constant if there is an eigenvector with eigenvalue 1 appearing with nonzero constant in the decomposition of x_0. So $\boxed{\text{decaying and going to a nonzero constant should be circled, and nothing else.}}$

(d) For $x_1 = Ax_0$ to be zero with x_0 nonzero means that $x_0 \in N(A)$, so x_0 is an eigenvector for A with eigenvalue 0. This can only happen if $(2-\lambda)/(2+\lambda) = 0$ for some eigenvalue λ of $B^T B$, so $B^T B$ must have an eigenvalue equal to 2. The singular values of B are the positive square roots of the positive eigenvalues of $B^T B$, so $\boxed{\sqrt{2}}$ is a singular value of B.

Problem 3 (15 points):

The distance between a point b and a plane in $\mathbb{R}^3$ is defined as the *minimum* distance $\|b - y\|$ between b and *any* point y in the plane.

(a) Suppose the points y in the plane are of the form $y = c + \alpha a_1 + \beta a_2$ for all real numbers α and β, given vectors $c, a_1, a_2 \in \mathbb{R}^3$ that define the plane (a_1 and a_2 are linearly independent). Under **what condition(s)** on c, a_1, a_2 is the plane a **subspace** of $\mathbb{R}^3$?

(b) Write down a 2×2 **system of equations**, in terms of the vectors a_1, a_2, c, b (or matrices defined from these vectors) whose solution gives the $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for the *closest* point y in the plane to b.

(c) For this closest point y, $b - y$ is in **what subspace** of the matrix $A = (a_1 a_2)$? What is the **dimension** of this subspace?

(d) For $a_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, **find a vector** d such that the distance between any point b and the plane is equal to $|d^T(b-c)|$. **What subspace** of A contains d?

Solution:

(a) Subspaces have to contain the zero vector, so there have to be real numbers α_c, β_c such that $c + \alpha_c a_1 + \beta a_{c2} = 0$. But then $c = -\alpha_c a_1 - \beta_c a_2$

so $\boxed{c \in \operatorname{span}(a_1, a_2)}$. To see that this necessary condition is also sufficient, notice that in this case the points on the plane are those points $c + \alpha a_1 + \beta a_2 = (\alpha + \alpha_c)a_1 + (\beta + \beta_c)a_2$ where α, β are arbitrary real numbers, which is just the span of a_1 and a_2, which is indeed a subspace (although if a_1 and a_2 are dependent as well, the subspace is no longer a plane).

Note: While the question intended to ask for a condition on c, the original phrasing (which had omitted the words "on c, a_1, a_2") was somewhat ambiguous in hindsight, so we gave full credit to solutions that described only the general conditions for a subset of $\mathbb{R}^3$ to be a subspace.

(b) Take any point $y = c + \alpha a_1 + \beta a_2$ on the plane. Then its distance to b is

$$\|y - b\| = c + \alpha a_1 + \beta a_2 - b\| = \| \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - (b - c)\|$$

(notice that $\begin{pmatrix} a_1 & a_2 \end{pmatrix}$ is a 3 by 2 matrix with columns given by the 3 by 1 vectors a_1 and a_2). We want to choose α, β to make this distance (or its square) as small as possible, which is the usual least squares problem for $\|Ax - v\|^2$ where in this case $A = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and $v = b - c$. A vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ will describe a point $y = c + \alpha a_1 + \beta a_2$ minimizing the distance to b if and only if the associated normal equations for this least squares problem are satisfied, i.e.

$$A^T A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A^T(b - c) = \begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} (b - c).$$

Multiplying together the first two matrices on the left and right, this reads:

$$\boxed{\begin{pmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_1^T(b - c) \\ a_2^T(b - c) \end{pmatrix}}$$

which is a 2 by 2 system of equations in α, β, as needed.

(c) If y is the closest point on the plane to b, then $y - b$ must be orthogonal to the plane. But the plane is a translate of the column space $C(A)$ of A by the vector c, so it is parallel to $C(A)$, so $y - b$ is orthogonal to $C(A)$. So, $y - b \in C(A)^\perp = N(A^T)$, the $\boxed{\text{left nullspace of } A}$. As $C(A)$ is a plane and therefore has dimension 2, the left nullspace $N(A^T)$, its orthogonal complement in $\mathbb{R}^3$, has $\boxed{\text{dimension } 3 - 2 = 1}$.

(d) The vectors d that will do the job are either of the unit vectors perpendicular to $C(A)$ (recall that if d is such a vector then $dd^T(b - c)$ will be the orthogonal projection of $b - c$ to the line spanned by d perpendicular $C(A)$, whose length $|d^T(b-c)|$ is the length of the component of $b - c$ perpendicular from the plane, i.e. is the distance from b to the plane). So suppose $d = (d_1, d_2, d_3)$. That d is orthogonal to α_2 says that $0 = d_1 = d \cdot a_2$, so we need to have $d_1 = 0$. That d is orthogonal to a_1 then says that $2d_2 + d_3 = 0$, so d must be a unit vector parallel to the vector $(0, 1, -2)$. Rescaling this vector by its length, we arrive at an answer: $\boxed{d = \pm\sqrt{5}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}$. Such a vector d is orthogonal to $C(A)$, so $d \in C(A)^{\perp} = N(A^T)$, the $\boxed{\text{left nullspace of } A}$.

Problem 4 (15 points):

You are given the following matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 4 \\ 1 & -2 & 1 & -4 \end{pmatrix}$$

(a) Find the **complete solution** x (i.e. all solutions) to $Ax = b$ for $b = \begin{pmatrix} 3 \\ 9 \\ -4 \end{pmatrix}$.

(b) $A^T y = d$ is solvable if and only if $d^T z = 0$ for some z. **Give such a vector** z.

Solution:

(a) We can solve the equation by row-reducing the augmented matrix $(A|b)$:

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 2 & 1 & -3 & 4 & 9 \\ 1 & -2 & 1 & -4 & -4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & -2 & 2 & -5 & -7 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & -2 & 2 & -5 & -7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right)$$

which is now simple enough to solve directly. Notice that the 3 by 4 matrix on the left of the line has rank 3, so the nullspace of A must have dimension 4 - 3 $=$ 1 by rank-nullity, so the nullspace will be the span of any nonzero vector we can find in the nullspce. Looking for ways to combine the columns to get 0, we see that we can't use the 4th column at all due to its nonzero third entry, and then you can see that the sum of the first three columns is zero, so $N(A) = \text{span}(1, 1, 1, 0)$. To find a particular solution, we need to find a way to combine the columns of this last 3 by 4 matrix to form the column on the right. Looking at the last entry we see we have to take exactly one of the fourth column, and then can use two of the first column and one of the second colum to get (3,3,-1). So (2,1,0,1) is a particular solution. (We could also use backsubstitution on the pivot columns.) So the complete solution is:

$$\boxed{\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}}$$

i.e. the line through (2,1,0,1) parallel to (1,1,1,0).

(b) Recall that $A^T y = d$ is solvable if and only if $d \in C(A^T)$. As $C(A^T) = N(A)^\perp$ and $N(A) = \text{span}(1,1,1,0)$, we see that $\boxed{d = (1,1,1,0)}$ is a possible answer. So is any nonzero multiple of it (and these are the *only* correct answers)

Problem 5 (15 points):

QR factorization of the matrix A (e.g. via Gram–Schmidt) yields $A = QR$, where

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 0 \\ & 1 & 2 \\ & & 2 \end{pmatrix}.$$

(a) Which columns of A were **orthogonal** to begin with, if any?

(b) What is the orthogonal **projection** p of the vector $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $C(A)$?

(c) If we are minimizing $\|Ax - b\|$ (i.e. solving the least-square problem) for $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, you should be able to *quickly* **get an upper-triangular system** of equations $U\hat{x} = c$ for the least-square solution $\hat{x}$. **What are the** upper-triangular matrix U and the right-hand-side vector c?

Solution:

(a) Call the columns of Q q_1, q_2, q_3 and the columns of A a_1, a_2, a_3. The matrix factorization $A = QR$ with R the matrix given above then just means that $a_1 = q_1, a_2 = 2q_1 + q_2, a_3 = 2q_2 + 2q_3$. The q_i are orthonormal, so the dot products of the columns of A are the same as the dot products of the columns of R! Only the first and third columns of R are orthogonal, so $\boxed{\text{only the first and third}}$ columns of A are orthogonal to begin with.

Another way to see this is to recall that the entries in the upper triangle of R are precisely the dot products that occur in during Gram–Schmidt. There is only one zero entry in R, in the upper-right corner, so it was only the first and third columns that were orthogonal.

(b) As the q_i from (1) form an orthonormal basis for $C(A)$, the orthogonal projection of b onto $C(A)$ is

$$(q_1 q_1^T + q_2 q_2^T + q_3 q_3^T)b = (4/\sqrt{2})q_1 + (4/2)q_3 = (2,0,-2,0) + (1,1,1,1) = \boxed{(3,1,-1,1)}.$$

(c) The normal equations must be satisfied, i.e. $A^T A\hat{x} = A^T b$. As $A = QR$, we have $A^T A = R^T Q^T QR = R^T IR = R^T R$ because Q is orthogonal, so the normal equations simplify to $R^T R\hat{x} = A^T b = R^T Q^T b$. The matrix R^T is invertible, so we can cancel it from each side, giving $\boxed{R\hat{x} = Q^T b}$ (this is our upper-triangular system ... we actually derived this equation in class!). So $\boxed{U = R}$ and $\boxed{c = Q^T b}$ do the trick. Concretely, $U = R$ is given above, and $c = Q^T b$ is 4 times the transpose of the first row of Q, given by:

$$\boxed{Q^T b = \begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \\ 2 \end{pmatrix}}.$$

Problem 6 (15 points):

You are given the nonsymmetric, diagonalizable matrix

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & -3 & -2 \\ -1 & 0 & -3 \end{pmatrix}.$$

and we want to understand the solutions of the ODE

$$\frac{dx}{dt} = Ax$$

for some initial condition $x(0)$.

(a) Show (by any test you want, e.g. the pivot test) that the matrix $A + A^H = \begin{pmatrix} -2 & 2 & 2 \\ 2 & -6 & -2 \\ 2 & -2 & -6 \end{pmatrix}$ is **negative definite**.

(b) If $Av = \lambda v$ is an eigensolution of A (v and λ may be complex), look at $v^H \left(A + A^H\right) v$ and use the fact that $A + A^H$ is negative definite to **show** that the **real part** of λ must be **negative**.

(c) What can you conclude from the previous parts about the solutions $x(t)$ as $t \to \infty$?

(d) If $A + A^H$ is negative definite (so that A's eigenvalues have negative real parts), but A is **defective**, does your answer to the previous part about $x(\infty)$ **change? Why** or why not?

Solution:

(a) One approach, as suggested, is to use the *pivot test:* do elimination, and check that the **pivots are negative**. Gaussian elimination on A yields

$$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -6 & -2 \\ 2 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} -2 & 2 & 2 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

which has pivots $-2, -4, -4$ that are all negative, so A is indeed negative definite.

Alternatively, we could apply the determinant test to $-(A + A^H)$, by showing that the determinants of the 1×1, 2×2, and 3×3 upper-left submatrices are all positive: since $A + A^H$ being negative-definite is equivalent to $-(A + A^H)$ being positive-definite. (You could also apply a determinant test directly to $A + A^H$, but it is tricky: the determinants of odd-size submatrices should be *negative* and the determinants of even-size submatrices should be *positive*, because of what happens to the determinant when you flip the sign of a matrix. Yuck!) The 1×1 submatrix determinant is $2 > 0$. The 2×2, determinant is $2 \cdot 6 - (-2) \cdot (-2) = 8 > 0$. For the 3 by 3 determinant we can clean up the matrix by adding the first row to each of the second rows of $-(A + A^H)$, without changing the determinant, giving:

$$\begin{pmatrix} 2 & -2 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

whose determinant is the product of its diagonal entries (because it is triangular), which is $2 \cdot 4 \cdot 4 > 0$. Probably the pivot test is easier!

(b) As suggested, we compute

$$v^H(A+A^H)v = v^H(Av)+(v^H A^H)v = v^H(\lambda v)+(Av)^H v = \lambda\|v\|^2+\bar{\lambda}\|v\|^2 = (\lambda+\bar{\lambda})\|v\|^2 < 0.$$

As $A + A^H$ is negative definite, that number needs to be a negative real number. But $\|v\|^2$ is a positive real number, so this means that

$\lambda + \bar{\lambda} = 2\,\mathrm{Re}\,\lambda$ (twice the real part of λ) must be a negative real number, so $\mathrm{Re}\,\lambda$ is a negative real number itself, as needed.

In particular, note that the key step is $v^H A^H = (Av)^H = (\lambda v)^H = \bar{\lambda} v^H$.

(c) They always $\boxed{\text{decay to zero}}$, since $e^{\lambda t} \to 0$ when $\mathrm{Re}\,\lambda < 0$.

(d) $\boxed{\text{The answer does not change.}}$ Even if A is defective, the coordinates of $x(t) = e^{At}x(0)$ are sums of multiples of functions of the form $t^n e^{\lambda t}$ where λ is an eigenvalue of A and n is a nonnegative integer (smaller than the size of the largest Jordan block of A with eigenvalue λ). But no matter what such n is, $\lim_{t\to\infty} t^n e^{\lambda t} = 0$ as long as $\mathrm{Re}\,\lambda < 0$ (the exponential decay "wins" compared to polynomial growth).

Problem 7 (10 points):

The following parts can be **answered independently** (and refer to **different matrices**). Little or no calculation should be needed.

(a) If $C(B)$ is a subspace of $N(A)$, then either (**circle one**) AB or BA must be simply ______.

(b) If A is a real-symmetric 3×3 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and corresponding real eigenvectors v_1, v_2, v_3, then an explicit equation for $A^{-1}b$ in terms of sums/products involving these eigenvectors and b, with *no matrix inverses,* is:

____.

(c) If A is a 3×3 non-singular real matrix with singular values $\sigma_1, \sigma_2, \sigma_3$, then give formulas in terms of $\sigma_1, \sigma_2, \sigma_3$ for $\det(A^T A) =$____________ and $|\det(A)| =$____________.

(d) If $N(A)$ is spanned by the vector $v \neq 0$, then projection matrices onto *two* of the fundamental subspaces of A are:

______________________________ and ______________________________
(write down two matrices and indicate which subspaces they project onto).

(e) If A is similar to the matrix $\begin{pmatrix} 3 & 6 & 2 \\ & 17 & 3 \\ & & 4 \end{pmatrix}$, then the eigenvalues of A are: ____________________.

Solution:

(a) $\boxed{\mathrm{AB} = 0}$ (i.e. AB should be circled and "zero" should be written in the blank.) The reason is that for any vector x, $Bx \in C(B) \subseteq N(A)$, so $ABx = 0$. But the only way $ABx = 0$ for *all* x can be true is if $AB = 0$.

(b) As the eigenvalues are distinct and as A is real symmetric, the corresponding eigenvectors v_1, v_2, v_3 must be orthogonal. So also v_1, v_2, v_3 is an orthogonal basis (not necessarily orthonormal, although they could be rescaled to be orthonormal!) consisting of eigenvectors of A^{-1} with corresponding eigenvalues $1, 1/2, 1/3$, respectively. So the matrix A^{-1} is the sum of three rank-1 matrices (orthogonal projections onto the eigenvectors):

$$A^{-1} = \frac{v_1 v_1^T}{v_1^T v_1} + \frac{1}{2}\frac{v_2 v_2^T}{v_2^T v_2} + \frac{1}{3}\frac{v_3 v_3^T}{v_3^T v_3}$$

so the desired expression is

$$\boxed{\left(\frac{v_1 v_1^T}{v_1^T v_1} + \frac{1}{2}\frac{v_2 v_2^T}{v_2^T v_2} + \frac{1}{3}\frac{v_3 v_3^T}{v_3^T v_3}\right) b}.$$

(c) The eigenvalues of $A^T A$ are the squares of the signular values of A, and the determinant is the product of the eigenvalues, so $\boxed{\det(A^T A) = \sigma_1^2\sigma_2^2\sigma_3^2}$. The singular value decomposition of A is $A = U\Sigma V^T$ where U, V are orthogonal matrices and Σ is the diagonal matrix with diagonal entries $\sigma_1, \sigma_2, \sigma_3$. So $\det(A) = \det(U)\det(\Sigma)\det(V^T)$. As Σ is diagonal we have $\det(\Sigma) = \sigma_1\sigma_2\sigma_3$. The determinant of an orthogonal matrix is ± 1, so we see that $\boxed{|\det(A)| = \sigma_1\sigma_2\sigma_3}$. Alternatively, $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2$, so we again obtain $|\det(A)| = \sqrt{\det(A^T A)} = \sigma_1\sigma_2\sigma_3$.

(d) $N(A)$ is spanned by $v \neq 0$ so

$$\boxed{\text{the projection onto } N(A) \text{ is } \frac{vv^T}{v^T v}}.$$

The orthogonal complement of $N(A)$ is $N(A)^\perp = C(A^T)$, the rowspace of A. Recall that if S is a subspace and P is the orthogonal projection onto S, then $I - P$ is the orthogonal projection onto $S^\perp$. So, we see that

$$\boxed{\text{the orthogonal projection onto } C(A^T) \text{ is } I - \frac{vv^T}{v^T v}}.$$

(e) Similar matrices have the same eigenvalues. Triangular matrices have the same eigenvalues as they have diagonal entries, so the eigenvalues of A are $\boxed{3, 17, 4}$.