MIT 18.06 Exam 3, Fall 2017
Johnson

Your name: ______________________

Recitation: ______________________

problem	**score**
1	/30
2	/30
3	/40
total	/100

Problem 1 (30 points):

(a) **Give a matrix** A where $\det(A - \lambda I) = 0$ has exactly two roots $\lambda = 1$ and $\lambda = 3$, but the trace of A does *not* equal 4.

(b) The eigenvalues of $(A + A^T)^{-1}$ for any real, square matrix A (assuming $A + A^T$ is invertible) must be ________________________.

(c) If $A = Q^T \Lambda Q$ for a diagonal matrix Λ and a real orthogonal matrix Q, then the eigenvectors of A are the ________________________ of Q.

(d) If A is real, and $e^{At}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = e^{(3+4i)t}\begin{pmatrix} 1+i \\ 2-2i \end{pmatrix} + e^{\alpha t}\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, then the ($t$-independent) scalars α, β, γ are ____?

(e) If A is a 4×4 matrix with $\det A = 5$, then $\frac{d}{dt}\det(A^T A t) =$________________________.

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Problem 2 (30 points):

You are given a matrix $A = e^{-B^T B}$ for some real 3×3 matrix B. The nullspace $N(B)$ is spanned by $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) **Circle** any of the following vectors that *cannot possibly* be eigenvectors of A, and put a **rectangle** around any vectors that *must be* eigenvectors of A:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) $A^n x$ for some $x \neq 0$ *may* do what for large n (circle *all possibilities*)? **Oscillate** / **decay** / **diverge** / **go to a nonzero constant vector**.

(c) For $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, give a **good approximation** for $A^n x$ for a very large n.

(blank page for your work if you need it)

Problem 3 (40 points):

The vector $x(t)$ satisfies the ODE

$$(I + A)\frac{dx}{dt} = (A^2 - I)x$$

for the **diagonalizable** matrix $A = \begin{pmatrix} 0.9 & 0.0 & 0.3 \\ 0.0 & 0.8 & 0.4 \\ 0.1 & 0.2 & 0.3 \end{pmatrix}$. If we square this, we get $A^2 = \begin{pmatrix} 0.84 & 0.06 & 0.36 \\ 0.04 & 0.72 & 0.44 \\ 0.12 & 0.22 & 0.2 \end{pmatrix}$.

(a) If A has an eigenvalue λ and an eigenvector v, give **a nonzero solution** $x(t)$ satisfying the ODE above, in terms of λ, v, and t.

(b) Both A and A^2 are ________________________ matrices. By inspection of A^2, what can you say (with no arithmetic! don't calculate λ!) **about the magnitudes** $|\lambda|$ of the three eigenvalues of A^2? What does this tell you about the magnitudes $|\lambda|$ of the eigenvalues of A?

(c) Give the **eigenvalue** λ of A with the biggest magnitude. A corresponding **eigenvector** is $\begin{pmatrix} \alpha \\ 2 \\ 1 \end{pmatrix}$ for what α?

(d) For an initial conditions $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, circle what would you expect the solutions $x(t)$ to do for large t: **Oscillate** / **decay** / **diverge** / **go to a nonzero constant vector**? Give a good **approximation for** $x(t)$ for a large t — if you can't figure it out exactly, at least give a vector that $x(t)$ is nearly parallel to.

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