MIT 18.06 Exam 3 Solutions, Fall 2017
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Problem 1 (30 points):

(a) **Give a matrix** A where $\det(A - \lambda I) = 0$ has roots $\lambda = 1$ and $\lambda = 3$, but the trace of A does *not* equal 4.

(b) The eigenvalues of $(A + A^T)^{-1}$ for any real, square matrix A (assuming $A + A^T$ is invertible) must be ______________________.

(c) If $A = Q^T \Lambda Q$ for a diagonal matrix Λ and a real orthogonal matrix Q, then the eigenvectors of A are the ______________________ of Q.

(d) If A is real, and $e^{At}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = e^{(3+4i)t}\begin{pmatrix} 1+i \\ 2-2i \end{pmatrix} + e^{\alpha t}\begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, then the ($t$-independent) scalars α, β, γ are ____?

(e) If A is a 4×4 matrix with $\det A = 5$, then $\frac{d}{dt}\det(A^T At) =$______________________.

Solution:

(a) The trace is the sum of the roots, including multiplicity, so all that we need is a matrix where one of the roots is repeated. The easiest way to do this is with a diagonal matrix, for example:

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$$

Here, the characteristic polynomial is $(1-\lambda)(3-\lambda)^2$, which has roots 1 and 3, but the trace is 7.

In hindsight, the question wording was somewhat ambiguous. You could also read it as saying that $\lambda = 1$ and $\lambda = 3$ are *among* the roots, but are not *all* of the roots. In that reading, you just need to provide a matrix that has a third eigenvalue of any value. e.g. a diagonal matrix with $1, 2, 3$ on the diagonal.

(b) $A+A^T$ is real-symmetric, so $(A+A^T)^{-1}$ is real-symmetric as well. Therefore its eigenvalues are real. As it is also invertible, none of its eigenvalues are zero. Considering the 1 by 1 case, $A = (a)$ for $a \neq 0$, we see that $(A + A^T)^{-1} = 1/(2a)$ has eigenvalue $1/(2a)$, which can be any nonzero real number depending on how we choose a. So that's precisely the condition: $\boxed{\text{nonzero and real.}}$

(c) A basis consisting of eigenvectors of A appears as the $\boxed{\text{rows}}$ of Q, as it's written. The quick way to see this is that normally we write a diagonalization as $A = X\Lambda X^{-1}$ where the *columns* of X are eigenvectors. Here, since $Q^{-1} = Q^T$, we get the usual diagonalization with $X = Q^T$ and $X^{-1} = Q$. The columns of Q^T (the eigenvectors) are the *rows* of Q.

(d) A is real and $(2, 4)$ is real, so if one eigenvalue is complex (here, $3+4i$), the solution must be a sum of complex-conjugate pairs. So, $\boxed{\alpha = 3 - 4i, \ \beta = 1 - i, \ \gamma = 2 + 2i}$.

(e) We have $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 = 5^2 = 25$ so $\det(A^T At) = t^4 \det(A^T A) = 25t^4$ so

$$\boxed{\frac{d}{dt}\det(A^T At) = \frac{d}{dt}(25t^4) = 100t^3.}$$

Problem 2 (30 points):

You are given a matrix $A = e^{-B^T B}$ for some real 3×3 matrix B. The nullspace $N(B)$ is spanned by $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) **Circle** any of the following vectors that *cannot possibly* be eigenvectors of A, and put a **rectangle** around any vectors that *must be* eigenvectors of A:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) $A^n x$ for some $x \neq 0$ *may* do what for large n (circle *all possibilities*)? **Oscillate** / **decay** / **diverge** / **go to a nonzero constant vector**.

(c) For $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, give a **good approximation** for $A^n x$ for a very large n.

Solution:

First, let's collect some observations. Recall from homework that $N(B^TB) = N(B)$, so $N(B^TB)$ is also spanned by $(1, 2, 1)$, and hence 0 appears as an eigenvalue of B^TB with multiplicity 1. Remember also that B^TB is symmetric positive semidefinite, so $-B^TB$ is symmetric negative semidefinite, so it is diagonalizable and all of its eigenvalues are nonpositive real numbers. As we already know that 0 appears as an eigenvalue with multiplicity 1, so all other eigenvalues are strictly negative. Taking matrix exponentials exponentiates the eigenvalues and preserves diagonalizability, eigenvectors, and multiplicity of eigenvalues. So, putting all of this together, we see that e^{-B^TB} is diagonalizable, has one eigenvalue equal to 1 with corresponding eigenvector $(1, 2, 1)$, and all other eigenvalues in the interval (0, 1). Because the matrix is symmetric, all eigenvectors not on the line spanned by (1,2,1) must be orthogonal to it. So this gives the answer to part (a): the (nonzero) vectors on the line spanned by (1,2,1) must be eigenvectors, and any vector neither on that line nor perpendicular to it, or that is 0, cannot be (one can also check that this is as much as you can possibly say).

(a) The vectors that have to be eigenvectors (and should be rectangled) are the multiples of the nullspace:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

and the vectors that cannot be eigenvectors (and should be circled) are the ones that are not orthogonal to the above vectors (or are zero, which cannot be an eigenvector):

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) For any x, $\lim_{n\to\infty} A^n x$ will exist and will equal the orthogonal projection of x onto the line spanned by the vector $(1, 2, 1)$ (because this line consists of eigenvectors of eigenvalue 1, and all other eigenvalues are less than 1 in absolute value. So in general this will allow the sequence $A^n x$ to $\boxed{\text{decay}}$ or $\boxed{\text{go to a nonzero constant vector}}$, depending on whether x has a nonzero component in the (1,2,1) direction.

(c) $\lim_{n\to\infty} A^n x$ is the orthogonal projection of x onto the line spanned by

$(1,2,1)$. So this limit is:

$$\lim_{n\to\infty} A^n x = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} = \boxed{\begin{pmatrix} 1/6 \\ 2/6 \\ 1/6 \end{pmatrix}}$$

Problem 3 (40 points):

The vector $x(t)$ satisfies the ODE

$$(I + A)\frac{dx}{dt} = (A^2 - I)x$$

for the **diagonalizable** matrix $A = \begin{pmatrix} 0.9 & 0.0 & 0.3 \\ 0.0 & 0.8 & 0.4 \\ 0.1 & 0.2 & 0.3 \end{pmatrix}$. If we square this, we get $A^2 = \begin{pmatrix} 0.84 & 0.06 & 0.36 \\ 0.04 & 0.72 & 0.44 \\ 0.12 & 0.22 & 0.2 \end{pmatrix}$.

(a) If A has an eigenvalue λ and an eigenvector v, give **a nonzero solution** $x(t)$ satisfying the ODE above, in terms of λ, v, and t.

(b) Both A and A^2 are ________________________ matrices. By inspection of A^2, what can you say (with no arithmetic! don't calculate λ!) **about the magnitudes** $|\lambda|$ of the three eigenvalues of A^2? What does this tell you about the magnitudes $|\lambda|$ of the eigenvalues of A?

(c) Give the **eigenvalue** λ of A with the biggest magnitude. A corresponding **eigenvector** is $\begin{pmatrix} \alpha \\ 2 \\ 1 \end{pmatrix}$ for what α?

(d) For an initial conditions $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, circle what would you expect the solutions $x(t)$ to do for large t: **Oscillate** / **decay** / **diverge** / **go to a nonzero constant vector**? Give a good **approximation for** $x(t)$ for a large t — if you can't figure it out exactly, at least give a vector that $x(t)$ is nearly parallel to.

Solution:

(a) We can look for a solution pointing in the direction of v, i.e. a solution of the form $x(t) = c(t)v$ for some not-always-zero scalar valued function

$c(t)$. Notice that for this $x(t)$, $(I+A)x'(t) = (I+A)c'(t)v = (1+\lambda)c'(t)v$ and $(A^2-I)x(t) = (A^2-I)c(t)v = c(t)(A^2-I)v = (\lambda^2-1)c(t)v$. So, this $x(t)$ will satisfy the given ODE if and only if $(1+\lambda)c'(t)v = (\lambda^2-1)c(t)v$ for all t. As A is invertible we must have $\lambda \neq 0$, so this means that we need $c'(t) = (1+\lambda)^{-1}(\lambda^2-1)c(t)$, which simplifies because $(\lambda^2-1) = (\lambda-1)(\lambda+1)$. We recognize a solution to this ODE given by $c(t) = e^{(\lambda-1)t}$. So this means that

$$\boxed{x(t) = e^{(\lambda-1)t}v}$$

gives a solution as needed.

(Note that we'll run into a problem if we have an eigenvalue $\lambda = -1$, since then we would have $1+\lambda = 0$. Fortunately, in the next part we show that this is not possible.)

(b) By inspection, both A and A^2 are $\boxed{\text{Markov}}$ matrices since the columns of A clearly sum to 1. In fact A^2 is a $\boxed{\text{positive Markov matrix}}$ (positive entries) so it has a $\boxed{\text{1-dimensional space of steady state vectors}}$ (= eigenvectors with eigenvalue 1) and $\boxed{\text{all other eigenvalues are less than 1 in absolute value}}$. As the eigenvalues of A^2, counted with multiplicity, are the squares of the eigenvalues of A it follows that the same statements hold for the eigenvalues of A.

It turns out that this particular A turns out to have purely real, positive eigenvalues, but it's not so easy to see that without more extensive calculations..

(c) $\boxed{\lambda = 1}$ is the largest-magnitude eignevalue, from above. To find a corresponding eigenvector, we should look for a vector in the nullspace of

$$A - I = \begin{pmatrix} -0.1 & 0.0 & 0.3 \\ 0.0 & -0.2 & 0.4 \\ 0.1 & 0.2 & -0.7 \end{pmatrix},$$

which is the same thing as looking for a nontrivial linear combination of the columns that gives 0. If (x,y,z) is such a vector, the first row says that $x = 3z$ and the second row says that $y = 2z$, so we see that the vector (3,2,1) is in the null space. So $\boxed{\alpha = 3.}$

(d) Very similar to one of the homework problems, $e^{(\lambda-1)t}$ is a constant if $\lambda = 1$ and is exponentially decaying for $|\lambda| < 1$ (where $\lambda - 1$ must have a negative real part). So, if we expand the initial vector in the basis of the eigenvectors (possible since A is given to be diagonalizable), then $x(t)$ is one constant term plus two decaying terms. So $x(t)$ will $\boxed{\text{limit to a nonzero constant vector}}$. This vector should be $\boxed{\text{in the direction of } \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}$, the eigenvector of $\lambda = 1$.

To determine the exact vector, similar to homework, we can easily show that the *sum of the components of $x(t)$ is conserved*. In particular, recall that $o^T A = o^T$ because A is a Markov matrix (here o is the vector with all entires equal to 1). If we multiply both sides of the differential equation by o^T, we get

$$o^T(I + A)\frac{dx}{dt} = 2o^T\frac{dx}{dt} = o^T(A^2 - I)x = (o^T - o^T)x = 0,$$

so $o^T\frac{dx}{dt} = \frac{d}{dt}(o^T x) = 0$, hence $o^T x$ is conserved. So the vector we're looking for is in the direction of $(3, 2, 1)$ and has sum of coefficients equal to the sum of coefficients of $x(0) = (1, 0, 0)$, i.e. 1. It follows that

$$\boxed{\lim_{t\to\infty} x(t) = \begin{pmatrix} 3/6 \\ 2/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}.}$$

Note that this problem could be simplified even further if you notice at the beginning that $(I + A)^{-1}(A^2 - I) = (A - I)$, since $A^2 - I = (A + I)(A - I) = A^2 + IA - AI - I^2$, but you didn't need this to solve the problem. (This approach relies on $I + A$ being nonsingular. Fortunately, from part b above, A cannot have an eigenvalue -1 so $I + A$ is indeed invertible.)