18.06 Professor Edelman Final Exam December 20, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Andrey Grinshpun	2-349	3-7578	`agrinshp`
2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	`agrinshp`
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	`robr`
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	`ghorel`
6	T 1	2-132	Tiankai Liu	2-491	3-4091	`tiankai`
7	T 2	2-132	Tiankai Liu	2-491	3-4091	`tiankai`

1 (10 pts.)

What condition on b makes the equation below solvable? Find the complete solution to $\mathbf{x}$ in the case it is solvable.

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}.$$

Solution:

Let's use Gaussian elimination. Starting from

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix},$$

multiply both sides by the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left, which has the effect of subtracting twice the first row from the second:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix}.$$

Then multiply both sides by the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ on the left, which has the effect of subtracting the second row from the third:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ b-1 \end{pmatrix}.$$

Comparing the third row on both sides, we find that $0 = b - 1$. The first and second rows of the matrix on the left-hand side both have pivots, so there are no other restrictions. The equation is solvable precisely when $b = 1$. Let us solve it in this case.

The free variables are x_2 and x_4. To find a *particular solution* to the equation at hand, set both free variables to zero, and solve for the pivot variables; we get $\mathbf{x} = (\frac{1}{2}, 0, \frac{1}{2}, 0)$. To find the complete solution, we must solve the homogeneous equation

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

The two special solutions to this homogeneous equation are found by setting one of the free variables to 1, the other to 0: we get $(-3, 1, 0, 0)$ and $(0, 0, -2, 1)$. Therefore, the complete solution to the original equation when $b = 1$ is

$$\mathbf{x} = \begin{pmatrix} \frac{1}{2} - 3x_2 \\ x_2 \\ \frac{1}{2} - 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

2 (6 pts.)

Let C be the cofactor matrix of an $n \times n$ matrix A. Recall that C satisfies $AC^T = (\det\ A)I_n$. Write a formula for $\det C$ in terms of $\det A$ and n.

Solution:

Since $AC^T = (\det\ A)I_n$, we get $\det(AC^T) = \det((\det\ A)I_n)$. The left hand side simplifies to $\det A \det C$ and the right hand side is equal $(\det A)^n$. This gives $\det C = (\det A)^{n-1}$.

The conscientious may object that we have divided both sides of the equation $\det A \det C = (\det A)^n$ by $\det A$, which is invalid if $\det A = 0$. So we still have to prove that, if $\det A = 0$, then C must also be singular. Well, assume for the sake of contradiction that $\det A = 0$ but C is invertible. Then C^T is also invertible, and we may multiply the original equation $AC^T = (\det A)I_n$ by $(C^T)^{-1}$:

$$A = (\det A)(C^T)^{-1} = 0(C^T)^{-1} = 0.$$

So A is the zero matrix, but in this case obviously so is its cofactor matrix C. This contradiction shows that indeed $\det C = (\det A)^{n-1}$ in all cases.

3 (25 pts.)

The matrix $A = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$ satisfies $A^2 = 6A$.

(a) (4 pts.) The eigenvalues of A are $\lambda_1 =$______, $\lambda_2 =$______, and $\lambda_3 =$______.

(b) (5 pts) Find a basis for the nullspace of A and the column space of A.

(c) (16 pts.) Circle all that apply. The matrix $M = \frac{1}{6}A$ is

1.orthogonal   3. a projection   5. singular   7. a Fourier matrix

2. symmetric   4. a permutation   6. Markov   8. positive definite

4 (12 pts.)

The matrix $G = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2-i & -1 & i \\ 1 & -1 & -1 & -1 \\ 1 & i & -1 & -2-i \end{pmatrix}$, where $i = \sqrt{-1}$.

(a) (6 pts) Use elimination or otherwise to find the rank of G.

(b) (6 pts) Find a real nonzero solution to $\frac{d}{dt}x(t) = Gx(t)$.

5 (12 pts.)

Given a vector x in $\mathbb{R}^n$, we can obtain a new vector $y = \text{cumsum}(x)$, the cumulative sum, by the following recipe:

$$y_1 = x_1$$
$$y_j = y_{j-1} + x_j, \text{ for } j = 2, ..., n.$$

(a) (7 pts) What is the Jordan form of the matrix of this linear transformation?

Solution:

Let's first find the matrix A representing the linear transformation cumsum, and then worry about finding the Jordan form of A. Note that cumsum maps $(1, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$, so the first column of A should be $(1, 1, \ldots, 1)$. The other columns of A can be found in a similar way, and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

If $n = 1$, then A is already in Jordan form. *So, for the rest of this solution, let's assume* $n \geqslant 2$.

To find the Jordan form of A, let's first find its eigenvalues. We see that A is a lower triangular matrix, so its eigenvalues are just its diagonal entries, which are all equal to 1. Thus, 1 is the only eigenvalue of A, and it occurs with arithmetic multiplicity n. This fact alone is not enough to determine the Jordan form of A, however. In fact, there are $p(n)$ non-similar $n \times n$ matrices whose only eigenvalue is 1, where $p(n)$ denotes the number of *partitions* of n — the number of distinct ways of writing n as a sum of positive integers, if order is irrelevant. Of the $p(n)$ possible distinct Jordan forms of such matrices, which one is actually the Jordan form of A?

One possibility that we can immediately eliminate is the identity matrix I_n. The only matrix similar to I_n is I_n itself: $MI_nM^{-1} = I_n$ for any invertible matrix M. Since A isn't the identity matrix, it isn't similar to I_n either, and its Jordan form is not I_n.

The key is to consider $A - I_n$: this matrix has rank $n-1$, because (for example) its transpose is in row-echelon form with $n-1$ pivots. Thus, $A - I_n$ has a 1-dimensional kernel, which is to say A has a *1-dimensional eigenspace* for the eigenvalue 1. This means that the Jordan form of A consists of a single Jordan block, which must therefore be

$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}$$

(the empty entries are zeroes).

(b) (5 pts) For every n,find an eigenvector of cumsum.

Solution:

An eigenvector for A is the same thing as a vector in the nullspace of

$$A - I_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}.$$

Row operations or mere inspection quickly leads to the conclusion that only the nth column is free, while all other columns have pivots. So all eigenvectors for A are scalar multiples of $(0, \ldots, 0, 1)$. It is easy to check that $(0, \ldots, 0, 1)$ is unchanged by the transformation cumsum, so this makes sense.

6 (20 pts.)

This problem concerns matrices whose entries are taken from the values +1 and -1. In other words, the general form of this matrix is $\begin{pmatrix} \pm 1 & \pm 1 & ... & \pm 1 \\ \pm 1 & \pm 1 & ... & \pm 1 \\ \vdots & \vdots & \ddots & \\ \pm 1 & \pm 1 & ... & \pm 1 \end{pmatrix}$. We will call these matrices ± 1matrices.

One $3x3$ example of such a matrix is $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

a) (5 pts.) Find a two by two example of a ± 1 matrix with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$ or prove it is impossible.

b) (5 pts.) Suppose A is a 10×10 example of a ± 1 matrix. Compute $\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{10}^2$

c) (5 pts.) The big determinant formula for a 5×5 A has exactly _________ terms. A computer package for matrices computes that the determinant of a ± 1 matrix that is 5×5 is an odd integer. If this is possible exhibit such a ± 1 matrix, if not argue clearly why the package must not be giving the right answer for this 5×5 matrix.

d) (5 pts.) For every n, construct a ± 1matrix $A_n$with $(n-1)$ eigenvalues exactly equal to 2. (Hint: Think about $A_n - 2I$.)

7 (15 pts.)

Let V be the six dimensional vector space of functions $f(x, y)$ of the form $ax^2 + bxy + cy^2 + dx + ey + f$. Let W be the three dimensional vector space of (at most) second degree quadratics in x.

a) (6 pts.) Write down a basis for V and a basis for W.

Solution:

A basis for V is $1, x, y, xy, x^2, y^2$. A basis for W is $1, x, x^2$.

b (9 pts.) In your chosen basis, what is the matrix of the linear transformation from V to W that takes $f(x, y)$ to $g(x) = f(x, x)$?

Recall that the ith column of the matrix simply describes the image of the ith basis vector of V as a linear combination of the basis vectors of W. Therefore, the transformation is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$