Solution Set 3, 18.06 Fall '11

1. (a) Do problem 1 from 3.2.

Solution. We perform row operations on A to get (first step is subtracting the first row from the second, the second step is subtracting the second row from the third)

$$\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&3\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}.$$

We see that x_1, x_3 are the pivot variables and x_2, x_4, x_5 are the free variables.

Perform row operations on B to get (we are subtracting twice the second row from the third)

$$\begin{bmatrix}2&4&2\\0&4&4\\0&8&8\end{bmatrix} \rightsquigarrow \begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}.$$

The pivot variables are x_1, x_2, the free variable is x_3. □

(b) Compute the column space and the null space for these matrices.

Solution. The column space is spanned by the pivot columns. For A the pivot columns are the first and the third, so the column space is

$$C(A) = x_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + x_3 \begin{bmatrix}2\\3\\1\end{bmatrix}.$$

For B the pivot columns are the first and the second, so the column space is

$$C(B) = x_1 \begin{bmatrix}2\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}4\\4\\8\end{bmatrix} = x_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}1\\1\\2\end{bmatrix}.$$

To find the nullspace of A we seek special solutions of

$$\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix} x = 0.$$

These are (we plug one of the free variables to be 1, the other free variables to be 0 and solve the resulting system)

$$\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix}.$$

Therefore the nullspace of A is

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The procedure for B is similar. We seek special solutions of

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} x = 0.$$

As x_3 is the only free variable, a corresponding special solution is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore the nullspace of B is

$$x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

□

(c) Compute the rank of these matrices.

Solution. The rank is equal to the number of pivot columns. In both cases it is equal to 2. □

(d) For both of these matrices M, solve $Mx = b$, where $b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

Solution. To solve $Ax = b$ we first perform the same row operations on b as we did on A (as if b were part of an augmented matrix) to get

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

But the system

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

has no solutions (look at the last row).

Now perform the same process for B to get

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}.$$

The system

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

has no solutions as well. □

2. Do problem 9 from 3.2.

   (a) False. The square matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has both x_1 and x_2 for free variables.

   (b) True. This is because the nullspace of an invertible matrix is trivial: the equation $Ax = 0$ has only $x = 0$ for a solution (multiply by A^{-1} from the left to see this). Any free variable, however, leads to a nontrivial element of the nullspace.

   (c) True. No two pivots can be in the same column.

   (d) True. No two pivots can be in the same row.

3. What is the sum of the number of free and pivot variables for a $m * n$ matrix in general? The answer should just depend on m and n.

   Solution. It's n. This is because an $m * n$ matrix has n columns; if r of them are pivot columns then the other $n - r$ are the free columns. □

4. Do problem 1 from 3.3.

   Solution. (a) Correct. The number of nonzero rows in the reduced echelon matrix R is easily seen to be the same as the number of pivots.

   (b) Incorrect. For otherwise any square matrix had rank 0 which is absurd: the 1×1 matrix $\begin{bmatrix} 1 \end{bmatrix}$ has rank 1.

   (c) Correct. The number of columns minus the number of free columns equals the number of pivot columns which is the rank.

   (d) Incorrect. This is because the rank of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is ≤ 2 (in fact $= 1$) although it has two 1's. □

5. Do problem 8 from 3.3.

Solution. Rank is the dimension of the column space. Therefore, for the rank to be 1 any two nonzero columns must be scalar multiples of each other. This reasoning allows us to recover A and B unequivocally:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}.$$

The same reasoning also works for M, except that we have to be careful and consider all posible cases:

- If $b \neq 0$ and $a \neq 0$ then a/b is what we are multiplying the second column to get the first. This gives $M = \begin{bmatrix} a & b \\ c & cb/a \end{bmatrix}$.
- If $b \neq 0$ but $a = 0$ and $c \neq 0$. In this case it is impossible to fill out $M = \begin{bmatrix} 0 & b \\ c & \end{bmatrix}$ so that it has rank 1. This is because both columns are nonzero and neither is a multiple of the other one.
- If $b \neq 0$ but $a = 0$ and $c = 0$. In this case any real number d will fill out M to $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ which is of rank 1 (because $b \neq 0$).
- If $b = 0$ and $a \neq 0$ then the first column is nonzero and 0 is what we have to multiply it by to get the second one. The only way to fill out M to a matrix of rank 1 is $M = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$.
- If $b = 0$ and $a = 0$ but $c \neq 0$. In this case any real number d gives $M = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ of rank 1.
- If $b = 0$ and $a = 0$ and $c = 0$. In this case we must pick $d \neq 0$ and any such will give $M = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ of rank 1. □

6. Do problem 13 from 3.4.

Solution. (a) If this were true then in particular $2x_p$ were a solution to $Ax = b$. But $A(2x_p) = 2(Ax_p) = 2b \neq b$ if $b \neq 0$ and we see immediately that the claim cannot be true.

(b) For the system $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ both $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are particular solutions.

(c) Consider the system $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As is seen by subtracting the first row from the second, x_2 is the free variable. Setting $x_2 = 0$ leads to the solution $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which has length 1. But the solution $x = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ has length $\sqrt{2}/2$ and is therefore shorter.

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible and yet has $x_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a solution for $Ax = 0$ in its nullspace. □

7. Fill a n by n matrix with random bits (i.e. either 1 or 0) and calulate its rank. In MATLAB this is `r = rank(randi([0 1],n))`. Do this many times for $n = 1, 5, 10$ and 15. What do you observe for these 4 situations? (Taking an average over the many times would be best, but you can also do this by just looking. You don't have to show each of the "many times" in your print out)

   Solution. When $n = 1$ the matrix will have rank 1 with probability 1/2 (and rank 0 with probability 1/2 as well), since in this case the rank will depend on whether the single entry is 1 or 0. As n increases, however, the probability that the matrix is not of full rank decays exponentially to zero. For instance, for $n = 15$ you should observe that the matrix almost always has a full rank (= 15). [See MATLAB code] □

8. *You do not need to touch MATLAB, or even the computer, for this problem.* Here is what happens when one uses MATLAB 's rank command:

```
>> e=1e-15; a=[1+e 1;1  1]; rank(a)
ans =
     1
```

   (the first command just means that $e = 10^{-15}$) Show that this is not mathematically correct. Why do you think MATLAB produces this answer? (No need to read MATLAB documentations - a couple of sentences with a reasonable guess would suffice)

   Solution. For any $A = \begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$ with $\epsilon \neq 0$ the rank of A is 2. To see this perform row operations (multiply the first row by $\frac{1}{1+\epsilon}$ and subtract it from the second):

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+\epsilon & 1 \\ 0 & 1-\frac{1}{1+\epsilon} \end{bmatrix},$$

   and observe that both columns are pivot (as $1 + \epsilon \neq 1$).

   While attempting to calculate the rank MATLAB is performing some standard calculations (that ought to work for any input) and is rounding off intermediate results (computers usually deal with numbers up to fixed precision). This is where the ϵ gets lost. □

9. Do problem 16 from 3.5.

Solution. (a) Any such vector is a scalar multiple of $e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. As $\{e\}$ is a linearly independent set, it is a desired basis.

(b) I claim that

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

is a basis for the vector space $V = \{x \in \mathbf{R}^4 \colon x_1 + x_2 + x_3 + x_4 = 0\}$ (note that certainly $v_1, v_2, v_3 \in V$). As

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ -c_1 - c_2 - c_3 \end{bmatrix},$$

it is clear that $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ gives $c_1 = c_2 = c_3 = 0$, i.e., v_1, v_2, v_3 are linearly independent. It remains to see why any $x \in V$ is a linear combination of v_1, v_2, v_3. This is because

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 v_1 + x_2 v_2 + x_3 v_3$$

as $x_4 = -x_1 - x_2 - x_3$.

(c) The vector space in question can be described as

$$W = \{x \in \mathbf{R}^4 \colon x_1 + x_2 = 0, x_1 + x_3 + x_4 = 0\}.$$

I claim that its basis is

$$w_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Firstly, $w_1, w_2 \in W$. Neither of those nonzero vectors is a scalar multiple of the other and so they must be linearly independent. It remains to show that they span W. But any $x \in W$ can be written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 w_1 + x_4 w_2.$$

Here we are using that $x_2 = -x_1$ and $x_3 = -x_1 - x_4$.

(d) The column space of I all of $\mathbf{R}^4$ for which we can take the standard basis

$$e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

As I is invertible its nullspace is $\mathbf{Z} = \{0\}$ and has the unique basis, namely, the empty set $\emptyset$ (note that the claim that the zero vector forms the basis of $\mathbf{Z}$ is incorrect: the cardinality of any basis must equal the dimension of the space, which in our case is 0). □

10. Do Problem 26 from 3.5.

Solution. Let E_{ij} denote the 3×3 matrix which has a single nonzero entry equal to 1 in position (i, j). In all parts it is immediate to verify that the claimed matrices form a basis. The dimension is equal to the cardinality of the basis.

(a) One can take E_{11}, E_{22}, E_{33} for a basis of the diagonal 3×3 matrices. The dimension is 3.

(b) Any symmetric matrix A is uniquely determined by its entries a_{ij} with $i \leq j$ which can be specified arbitrarily. Therefore the matrices E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33} form a desired basis. The dimension is 6.

(c) Any skew-symmetric matrix must have zeroes on the diagonal (this is because $a_{ii} = -a_{ii}$). Except for that it is uniquely determined by its entries a_{ij} with $i < j$ which can be specified arbitrarily. Therefore the matrices E_{12}, E_{13}, E_{23} form a desired basis. The dimension is 3. □

MATLAB code

```
%%%%%%%%%%%%%
% Problem 7 %
%%%%%%%%%%%%%

 t=100000;
 v=zeros(t,1);
 n=1; for i=1:t, v(i)=rank(randi([0 1],n));end; mean(v)
ans = 0.499440000000000

 n=5; for i=1:t, v(i)=rank(randi([0 1],n));end; mean(v)
ans = 4.270880000000000

 n=10; for i=1:t, v(i)=rank(randi([0 1],n));end; mean(v)
ans = 9.689080000000001
```

```
 n=15; for i=1:t, v(i)=rank(randi([0 1],n));end; mean(v)
ans = 14.952970000000001

  % We see that as n increases a random n x n matrix of 0's and 1's has full
  % rank almost surely.
```