18.06 (Fall '11) Problem Set 10

This problem set is due Monday, November 28, 2011 at 4pm. The problems are out of the 4th edition of the textbook. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary("filename")` will start a transcript session, `diary off` will end one.)

1. Do Problem 20 from 6.5.

    Solution. (a) *A positive definite matrix is invertible*: Because it has positive eigenvalues, so zero is not an eigenvalue.

    (b) *The only positive definite projection matrix is $P = I$ because*: Let V be the subspace that P projects on, i.e. the range of the map P. If its orthogonal complement $V^\perp = \{0\}$, then $P = I$. If $V^\perp \neq \{0\}$, then any non-zero element $v \in V^\perp$ satisfies $Pv = 0$, $v \neq 0$ and hence P is not invertible. Then see (a).

    (c) *There are two reasons a diagonal matrix with positive diagonal entries is positive definite*, not to forget: 1) Diagonal matrices are symmetric, and 2) The eigenvalues of a diagonal matrix are the diagonal entries.

    (d) *A symmetric matrix with a positive determinant might not be positive definite*: For example the 2×2 example $\det(-I_2) = (-1)^2 \det(I_2) = 1 > 0$ shows this.

    □

2. Compute the *cube* root (i.e. find D such that $D^3 = A$) for the positive definite symmetric square matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

    Solution. We diagonalize this symmetric matrix A and get:

    $$A = M\Lambda M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

    [Note that for our application, needing a cube root D such that $D^3 = A$ it is not necessary to divide by $\sqrt{2}$'s to make the M into an orthogonal Q]. Namely, we see that $(M\Lambda^{1/3}M^{-1})^3 = M\Lambda M^{-1} = A$. So we just get:

    $$D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9^{1/3} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{9^{1/3}+1}{2} & \frac{9^{1/3}-1}{2} \\ \frac{9^{1/3}-1}{2} & \frac{9^{1/3}+1}{2} \end{bmatrix}.$$

    □

3. Do Problem 5 from 6.6.

    Solution. We first list the eigenvalues of all the matrices, they are respectively: $(1, 1)$, $(1, -1)$, $(0, 1)$, $(0, 1)$, $(0, 1)$ and $(0, 1)$. Since similar matrices have the same eigenvalues

(with same algebraic and geometric multiplicities too), the only ones in this list that have a chance of being similar are the last four matrices.

Now, since the last four each have two different eigenvalues, we know they are all diagonalizable. The diagonal matrix is the same Λ, with diagonals 0 and 1. Since the last four are thus all similar to the same matrix Λ, all four are similar to each other. $\qquad\square$

4. Do Problem 17 from 6.6.

   Solution. (a) False, a symmetric matrix can easily be similar to a nonsymmetric one. Here's a pair of good reasons:

$$TAT^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   (b) True. If a matrix A is similar to a matrix B, then $\det A = \det B$. So A is singular if and only B is singular.

   (c) False. $-A = TAT^{-1}$ happens precisely when $AT + TA = 0$ (given that T is invertible).

   So, as a good reason we should find such a pair that "anti-commute". Famous examples are the Pauli matrices from Quantum Mechanics[1], usually denoted by σ's. Here are two of them (and they are invertible):

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{with} \quad \sigma_1\sigma_3 + \sigma_3\sigma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (d) True, A can never be similar to $A + I$. Remember that the trace is also an invariant under similarity since $\text{tr}(TAT^{-1}) = \text{tr}((AT^{-1})T) = \text{tr}(A)$ follows from the $\text{tr}(AB) = \text{tr}(BA)$ property. But if A is $n \times n$ then:

$$\text{tr}(A + I_n) = \text{tr}(A) + \text{tr}(I_n) = \text{tr}(A) + n \neq \text{tr}(A).$$

$\qquad\square$

5. Do Problem 22 from 6.6.

   Solution. **Jordan form:** The first hint says to show that $(J_n[0])^n = 0$, where J_n is a Jordan block of size n of eigenvalue 0. Recall that when we take powers of such a matrix, the non-zero elements "move further into the top right corner".

   To be more precise, as we must, let's introduce a notion: Let k be some integer. Assume an $n \times n$ matrix A has the property that $A_{ij} = 0$ whenever $i + k > j$. Then let's invent the word that A is "k-cornered". Notice that an

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[1]Historically, a discipline initially known as "Matrix Mechanics"!

upper diagonal matrix is 0-cornered, and that if an $n \times n$ matrix is k-cornered for some $k \geq n$, then it is truly cornered up and must be the zero matrix [2].

<u>Claim</u>: If $k \geq 1$, the product of two k-cornered matrices A and B is $(k+1)$-cornered.

To prove it, we take two such A, B and compute their product:

$$(AB)_{ij} = \sum_{s=1}^{n} A_{is} B_{sj}.$$

Fix n, k and i, j. Is it true, as the claim says, that $(AB)_{ij} = 0$ if the condition $i + k + 1 > j$ holds?

Indeed it is, for using that $A_{is} = 0$ when $i+k > s$ and $B_{sj} = 0$ when $s+k > j$, we see that some term $A_{is_0} B_{s_0 j}$ (happening for some $s_0\{1, \ldots, n\}$) can be non-zero only if both $i+k \leq s_0$ and also $s_0 + k \leq j$. But combining these two requirements gives $i + k \leq j - k \leq j - 1$, using $k \geq 1$. But this says that $i + k + 1 \leq j$, violating the condition $i + k + 1 > j$ we were assuming for these fixed i, j, k. Thus no such s_0 may happen, and therefore the whole sum is also zero for these i, j: $(AB)_{ij} = 0$ when $i + k + 1 > j$.

<u>Consequence of claim</u>: A Jordan block J_n of size n and with eigenvalue 0 satisfies $(J_n)^n = 0$. Namely, J_n is 1-cornered, $(J_n)^2$ is 2-cornered etc., so $(J_n)^n$ is an n-cornered $n \times n$ matrix, hence is the zero matrix.

Now, let A be any matrix with all eigenvalues being $\lambda_i = 0$. Let $s \in \{1, \ldots, n\}$ denote the number of independent eigenvectors. We then use Jordan's theorem $A = MJM^{-1}$, where

$$J = \begin{bmatrix} J_{n(1)} & & \\ & \ddots & \\ & & J_{n(s)} \end{bmatrix},$$

and each block is a Jordan block of size $1 \leq n(i) \leq n$, $i = 1, \ldots, s$ of eigenvalue 0. Since

$$J^n = \begin{bmatrix} (J_{n(1)})^n & & \\ & \ddots & \\ & & (J_{n(s)})^n \end{bmatrix} = 0,$$

and $A^n = MJ^nM^{-1}$, we are done.

Cayley-Hamilton: Since all n eigenvalues are 0, the characteristic polynomial must be $P(\lambda) = (\lambda - 0)^n = \lambda^n$. By Cayley-Hamilton, $P(A) = 0$, so $A^n = 0$.

□

6. What are the singular values of an n by n Jordan block with eigenvalue 0? In MAT-LAB A = gallery('jordbloc', n, e) creates a Jordan block of size n and eigenvalue e.

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[2]Since then $i + k \geq i + n > j$ holds for every pair $i, j \in \{1, \ldots, n\}$, so every entry is zero.

Solution. Let J_n be a Jordan block with eigenvalue 0. In general we get:

$$(J_n)^T J_n = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Hence any Jordan block of size $n \geq 2$ and eigenvalue 0 has the n singular values $1, \ldots, 1, 0$ (it is not enough to just say "0 and 1"!). $\square$

7. Do Problem 6 from 6.7.

   Solution.
   $$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

   Diagonalizing $A^T A$, we get:

   $$A^T A = V \Lambda_1 V^T = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}^T,$$

   $$A A^T = U \Lambda_2 U^T \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T,$$

   where we have fixed a V, then chosen the signs of the columns in U in such a way that $AU = V\Sigma$ indeed works, where:

   $$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   $\square$

8. Do Problem 7 from 6.7.

   Solution. "That matrix" refers to Problem 6 from 6.7 above. The closest rank one matrix A_1 approximating A comes from the largest singular value, which is $\sqrt{3}$, as $\sqrt{3} u_1 v_1^T$. Thus

   $$A_1 = \sqrt{3} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 \end{bmatrix}$$

   $\square$

9. Let V be the function space of polynomials with basis $1, x, x^2, x^3, x^4$. What is the matrix M_t (it should depend on t) for the operator that sends $f(x)$ to $f(x+t)$? Show that M_t and M_{-t} are inverses.

Solution. A basis element x^n gets sent to

$$(x + t)^n = \sum_{k=0}^{n} \binom{n}{k} x^k t^{n-k},$$

by the Binomial Theorem. Thus if we index the matrix by $\{0, 1, \ldots 4\}$ corresponding to $\{x^0 \ldots, x^4\}$, we get

$$(M_t)_{ij} = t^{j-i} \binom{j}{i}.$$

Warning: Don't forget to get the order $i \leftrightarrow j$ correct, to avoid writing M_{ji} by mistake(!)

We get:
$$M_t = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 1 & 3t & 6t^2 \\ 0 & 0 & 0 & 1 & 4t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

TIP of the day (as every day): *Always double check*!

We compute easily:

$$M_t M_{-t} = M_t = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 1 & 3t & 6t^2 \\ 0 & 0 & 0 & 1 & 4t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (-t) & (-t)^2 & (-t)^3 & (-t)^4 \\ 0 & 1 & 2(-t) & 3(-t)^2 & 4(-t)^3 \\ 0 & 0 & 1 & 3(-t) & 6(-t)^2 \\ 0 & 0 & 0 & 1 & 4(-t) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I.$$

$\square$

10. Let V be the function space with basis $\sin(x), \cos(x), \sin(2x), \cos(2x)$. What is the matrix of the derivative operator in this basis? What is its determinant? Why is the determinant what you get (you don't have to turn this part in, but do the mental exercise)?

Solution. We have

$$(\sin x)' = \cos(x), \quad \text{i.e.} \quad [1, 0, 0, 0] \mapsto [0, 1, 0, 0] = c_1$$
$$(\cos(x))' = -\sin(x), \quad \text{i.e.} \quad [0, 1, 0, 0] \mapsto [-1, 0, 0, 0] = c_2$$
$$(\sin(2x))' = 2\cos(2x), \quad \text{i.e.} \quad [0, 0, 1, 0] \mapsto [0, 0, 0, 2] = c_3$$
$$(\cos(2x))' = -2\sin(2x), \quad \text{i.e.} \quad [0, 0, 0, 1] \mapsto [0, 0, -2, 0] = c_4.$$

$\square$

Here we have written out where each basis vectors gets sent to. So, with respect to this basis we must insert these vectors as the *columns* c_i^T (remember how a matrix acts from the left: on each column),

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Q: Can you see why the following, while tempting to write and somewhat productive, is nonsensical and furthermore gives you the *wrong* matrix (namely the transpose D^T), and hence has been crossed out by your friendly, but insisting, TA?

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \sin(x) \\ \cos(x) \\ \sin(2x) \\ \cos(2x) \end{bmatrix} = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ 2\cos(2x) \\ -2\sin(2x) \end{bmatrix}.$$

A: The functions *are* the vectors themselves (a vector with vector entries can surely make sense, but that's fancy way beyond 18.06).

The determinant is: $\det D = 4$. This is a general fact about a matrix of so-called block-diagonal form (if A, B are $n \times n$ matrices):

$$\det\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \det(A)\det(B),$$

so here $\det D = 1 \cdot 4 = 4$.

18.06 Wisdom. Try to think about everything you've learned in other classes, especially those without the number 18 in them, as a linear transformation. How many can you name? Once this clicks – congratulations. That's the wizard behind the curtain.