18.06 F10 Problem Set 9 Solutions Due Thursday 18 Nov.

1. Problem 12, section 6.6, p.361.

Solution: Say $M = [m_{ij}]$. Then the equation $JM = MK$ becomes

$$\begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}.$$

Hence $m_{11} = m_{22} = 0$ and $m_{21} = 0$ and $m_{31} = m_{42} = 0$ and $m_{41} = 0$. In other words, the first column of M vanishes. Thus M is not invertible, as asserted.

2. Problem 13, section 6.6, p.361.

Solution: The six Jordan forms include the two, J and K, given in problem 12, the zero matrix and the following three matrices:

$$\begin{bmatrix} 0 & 0 & & \\ & 0 & 0 & \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & & \\ & 0 & 0 & \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

(Problem 12 shows that J and K are inequivalent forms, although both have two 1 s above the diagonal.)

3. Problem 1, section 6.7, p.371.

Solution: As $rk(A^{\mathrm{T}}A) = 1$, we have $\sigma_2 = 0$ and $\sigma_1^2 = tr(A^{\mathrm{T}}A) = 10 + 40 = 50$. So $A^{\mathrm{T}}A - \sigma_1^2 I$ is $\left[\begin{smallmatrix} -40 & 20 \\ 20 & -10 \end{smallmatrix}\right]$. A nullvector is $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. So $v_1 = \frac{1}{\sqrt{5}} \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. An orthogonal vector is $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$. So $v_2 = \frac{1}{\sqrt{5}} \left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$. (As a check, note that v_1 spans the row space $C(A^{\mathrm{T}})$ and v_2 spans the nullspace $N(A)$.) Then $u_1 = Av_1/\sigma_1 = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] / (\sqrt{5} \cdot \sqrt{50}) = \left[\begin{smallmatrix} 5 \\ 15 \end{smallmatrix}\right] / 5\sqrt{10} = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right] / \sqrt{10}$. Further, $||u_i|| = \sqrt{1+3^2}/\sqrt{10} = 1$, and $A^{\mathrm{T}}Au_1 = \left[\begin{smallmatrix} 5 & 15 \\ 15 & 45 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right] \left[\begin{smallmatrix} 50 \\ 150 \end{smallmatrix}\right] / \sqrt{10} = \left[\begin{smallmatrix} 50 \\ 150 \end{smallmatrix}\right] / \sqrt{10} = \sigma_1^2 u_1$. Finally, the SVD of A is this:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \left(\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \Big/ \sqrt{10} \right) \begin{bmatrix} \sqrt{50} & \\ & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \Big/ \sqrt{5} \right).$$

4. Problem 2, section 6.7, p.372.

Solution: The four fundamental subspaces are each 1-dimensional. So orthonormal bases are formed by the four vectors v_1, v_2, u_1, u_2. Explicitly, these basis vectors are as follows:

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} / \sqrt{5} \text{ for } C(A^{\mathrm{T}}); \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} / \sqrt{5} \text{ for } N(A);$$

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} / \sqrt{10} \text{ for } C(A); \quad u_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} / \sqrt{10} \text{ for } N(A^{\mathrm{T}}).$$

5. Problem 3, section 6.7, p.372.

Solution: As A has rank 1, so does $A^{\mathrm{T}}A$; in fact, if $A = uv^{\mathrm{T}}$, then $A^{\mathrm{T}}A = vu^{\mathrm{T}}uv^{\mathrm{T}} = ||u||^2 vv^{\mathrm{T}}$. Hence, $A^{\mathrm{T}}A$ has only one nonzero eigenvalue, and it is equal to the trace $tr(A^{\mathrm{T}}A)$. But the jth diagonal entry of $A^{\mathrm{T}}A$ is the dot product of the jth column with itself. So this entry is the sum $\Sigma_i\, a_{ij}^2$. Thus $\sigma_1^2 = tr\,(A^{\mathrm{T}}A) = \Sigma_{i,j} a_{ij}^2$.

6. Problem 6, section 6.7, p.372.

Solution: Since $A = \left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, we find

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad A\,A^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since $A^{\mathrm{T}}A$ is of larger size then AA^{T}. We use AA^{T} to find the σ_i. The characteristic polynomial of $A^{\mathrm{T}}A$ is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)\,(\lambda - 1)$. So $\boxed{\sigma_1^2 = 3 \text{ and } \sigma_2^2 = 1}$. Then $AA^{\mathrm{T}} - \sigma_1^2 I = \left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. So $\left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right] u_1 = 0$. Thus $\boxed{u_1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] / \sqrt{2}}$. Further, $AA^{\mathrm{T}} - \sigma_2^2 I = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. So $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] u_2 = 0$. Thus $\boxed{u_2 = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right] / \sqrt{2}}$. (As a check, note that u_1 and u_2 are orthogonal.) Nept,

$$v_1 = A^{\mathrm{T}}u_1/\sigma_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big/ (\sqrt{2} \cdot \sqrt{3}). \quad \text{So} \quad \boxed{v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Big/ \sqrt{6}}.$$

And

$$v_2 = A^{\mathrm{T}}u_2/\sigma_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Big/ (\sqrt{2} \cdot 1). \quad \text{So} \quad \boxed{v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Big/ \sqrt{2}}.$$

Further, v_3 is a nullvector of $A^{\mathrm{T}}A$. So by inspections,

$$\boxed{v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Big/ \sqrt{3}}.$$

Finally, the SVD of A is this:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Big/ \sqrt{2} \right) \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ 2 & 0 & -\sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{bmatrix}^{\mathrm{T}} \Big/ \sqrt{6} \right).$$

7. Problem 7, section 6.7, p.372.

Solution: The closest rank-one approximation to the matrix A in problem 6 is given by the first summand in the decomposition $A = \sum \sigma_i u_i v_i^{\mathrm{T}}$, namely,

$$\sigma_1 u_1 v_1^{\mathrm{T}} = \sqrt{3} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big/ \sqrt{2} \right) \left(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \Big/ \sqrt{6} \right) = \boxed{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} \Big/ 2}$$

8. Problem 8, section 6.7, p.372.

Solution: Since the rengular values of A^{-1} are the reciprocals of those of A, we have

$$\sigma_{\max}(A^{-1}) = 1/\sigma_{\max}(A) \ \geq \ 1/\sigma_{\max}(A).$$

Thus $\sigma_{\max}(A^{-1}) \ \sigma_{\max}(A) \geq 1$.

9. Problem 1, section 8.1, p.418.

Solution: At the top of p.410, the matrix $A_0^{\mathrm{T}} C_0 A_0$ is given explicitly. So

$$\det \ A_0^{\mathrm{T}} C_0 A_0 = \begin{vmatrix} c_1 + c_2 & -c_2 & \\ -c_2 & c_2 + c_3 & -c_3 \\ & -c_3 & c_3 + c_4 \end{vmatrix}.$$

Adding the first row to the second and then expanding along the first row, we get

$$\det \ A_0^{\mathrm{T}} \, C_0 \, A_0 = \begin{vmatrix} c_1 + c_2 & -c_2 & \\ c_1 & c_3 & -c_3 \\ & -c_3 & c_3 + c_4 \end{vmatrix}$$

$$= (c_1 + c_2) \begin{vmatrix} c_3 & -c_3 \\ -c_3 & c_3 + c_4 \end{vmatrix} - (-c_2) \begin{vmatrix} c_1 & -c_3 \\ & c_3 + c_4 \end{vmatrix}.$$

In the first 2 by 2 determinant, adding the first row to the second, we get

$$\begin{vmatrix} c_3 & -c_3 \\ -c_3 & c_3 + c_4 \end{vmatrix} = \begin{vmatrix} c_3 & -c_3 \\ & c_4 \end{vmatrix}.$$

Therefore, $\det A_0^{\mathrm{T}} C_0 A_0 = (c_1 + c_2)\, c_3\, c_4 + c_2\, c_1\, (c_3 + c_4) = c_1\, c_3\, c_4 + c_2\, c_3\, c_4 + c_1\, c_2\, c_3 + c_1\, c_2\, c_4$. Finally, setting $c_4 = 0$ yields $\det A_1^{\mathrm{T}} c_1\, A_1 = c_1\, c_2\, c_3$.