Problem 18 Section 6.1

Since A has 2 distinct eigenvalues it can be diagonalized:

$$A = S^{-1}\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} S$$

For $\quad S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$

$$S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad A_3 = S_3 \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} S_3^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$$

Problem 30 Section 6.1

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{if} \quad a+b = c+d = \lambda_1$$

$\text{tr } A = a + d = \lambda_1 + \lambda_2 \ \Rightarrow\ \lambda_2 = a + d - (a+b) = d - b (= a - c)$

The eigenvector corresponding to λ_2: $\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -c \end{pmatrix}$$

Problem 7 Section 6.2

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvectors $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

This means:

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this $\begin{pmatrix} a+b-\lambda_1 \\ c+d-\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a-b-\lambda_2 \\ c-d+\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Or $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{pmatrix}$ The solution: $a = d = \dfrac{\lambda_1+\lambda_2}{2}$, $c = b = \dfrac{\lambda_1-\lambda_2}{2}$

Thus the general form of the matrix is $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with a, b any number.

Problem 11   Section 6.2

If A is 3×3 then

(a) true: $\det A = 2 \cdot 2 \cdot 5 = 20 \neq 0$

(b) false: $A = \begin{pmatrix} 2 & 1 & \\ & 2 & \\ & & 5 \end{pmatrix}$

(c) false: $A = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 5 \end{pmatrix}$

Problem 7   Section 6.3

$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad P = \dfrac{a^{\mathrm{T}}\, a}{a\, a^{\mathrm{T}}} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

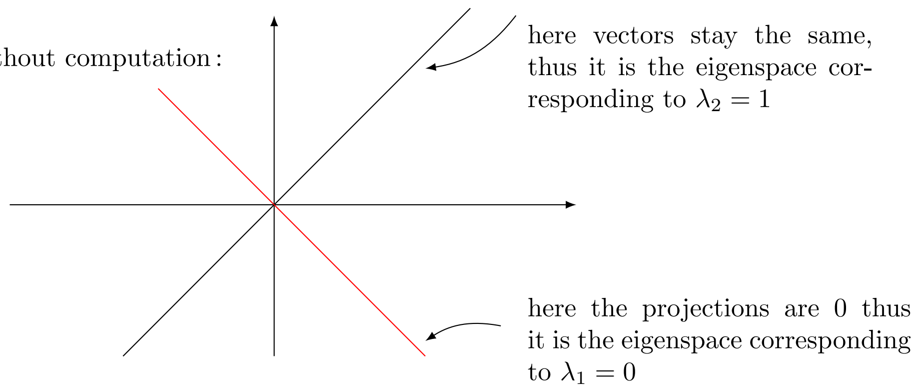
One can directly compute the eigenvalues by solving $\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$

This gives $\lambda_1 = 0$ and $\lambda_2 = 1$, and the eigenvectors

for $\lambda_1 = 0 \qquad \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

for $\lambda_2 = 1 \qquad \dfrac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

On without computation:



$\dfrac{d\,\underline{u}(t)}{dt} = -P\,\underline{u}(t) \qquad \underline{u}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\,\underline{v}_1 + 1\,\underline{v}_2$

We look for $\underline{u}(t)$ in the form $\qquad \underline{u}(t) = x(t)\underline{v}_1 + y(t)\underline{v}_2$

$-P\,\underline{u}(t) = -P\big(x(t)\,\underline{v}_1 + y(t)\,\underline{v}_2\big) = -x(t)\,\underline{v}_1 + 0$

$$\frac{d\,\underline{u}(t)}{dt} = \frac{d\,x(t)}{dt}\,\underline{v}_1 + \frac{d\,y(t)}{dt}\,\underline{v}_2$$

This means $\quad \dfrac{d\,x(t)}{dt} = -x(t) \qquad x(0) = 2$

$\qquad\qquad\quad \dfrac{d\,y(t)}{dt} = 0 \qquad y(0) = 1$

Solving the equations: $\left.\begin{array}{ccc} x(t) & = & 2e^{-t} \\ y(t) & = & 1 \end{array}\right\}$

Thus $\underline{u}(t) = 2e^{-t}\ \underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 2e^{-t}+1 \\ 2e^{-t}-1 \end{pmatrix}$

Problem 3 Section 6.6

- $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  eigenvalues: 1 and 0

  eigenvectors: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad \underline{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

  $\left.\begin{array}{l} \underline{u}_1 = \underline{v}_1 - \underline{v}_2 \\ \underline{u}_2 = \underline{v}_2 \end{array}\right\}$ Thus $M_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

- $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\qquad B_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

  eigenvalues: 2 and 0

  eigenvectors: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad \underline{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  $M_2 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ $\qquad \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

- $A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad B_3 = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

  eigenvalues: 2 and 0

  eigenvectors: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad \underline{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Problem 18 Section 6.6

$B\,(AB)B^{-1} = BA$

Problem 4 Section 8.3

Since the sum in each column is 1 we have: $A^{\mathrm{T}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Problem 11 Section 8.3

The sum in each column is 1, thus

$$A = \begin{pmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{pmatrix}$$

A is symmetric thus $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue 1. This means it is a steady state.