Fall 2010 18.06 Problem Set 7 Solutions

(1) (a) False, take $A = -I$ (2×2 matrices). Then $\det(I+A) = \det(0) = 0$ and $1+\det(A) = 2$.
(b) True, $\det(ABC) = \det(AB)\det(C) = \det(A)\det(B)\det(C)$.
(c) False, in general $\det(4A) = 4^n \det(A)$ if A is $n \times n$. For an explicit counterexample, take A to be the 2×2 identity matrix.
(d) False, take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AB - BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which has determinant 1.

(2) To turn J_n into the the identity matrix, we switch the first and last row with each other, the second and second-to-last row with each other, etc. If n is odd, the middle row will stay where it is, and if n is even, the middle two rows get switched with each other. So in total there are $\lfloor n/2 \rfloor$ row exchanges (rounding $n/2$ down if it is not a whole number). For $n = 5, 6, 7$, this becomes 2, 3, and 3 row exchanges, respectively. So J_n has determinant $(-1)^{\lfloor n/2 \rfloor}$, and in particular J_{101} has determinant 1.

(3) (a) True, remember that a matrix is invertible if and only if its determinant is nonzero. So $\det A = 0$ which means $\det(AB) = \det(A)\det(B) = 0$, hence AB can't be invertible.
(b) False, take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det A = -1$ but the product of pivots is 1.
(c) False, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = -A$. Then $\det(A - B) = 4$ and $\det A - \det B = 0$.
(d) True, $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$ (we can switch the order of the product of $\det(A)$ and $\det(B)$ because they are just numbers).

(4) (a) We calculate E_n by cofactors along the first row of the tridiagonal matrix of order n. There are only two nonzero entries in the first row, so we only need to figure out what those cofactors are. The first nonzero entry is the $(1, 1)$ position. Removing the first row and column, we are left with the tridiagonal matrix of order $n-1$. So this cofactor is E_{n-1}. The next nonzero entry is the $(1, 2)$ position. If we remove the first row and second column, we are left with some matrix T of size $(n - 1) \times (n - 1)$ that we can describe as follows. The first row of T is $(1, 1, 0, 0, \dots, 0)$. The first column of T is $(1, 0, 0, 0, \dots)^T$. The bottom right $(n - 2) \times (n - 2)$ submatrix of T is the tridiagonal matrix of size $n - 2$. So we can evaluate $\det T$ by cofactors along the first column to get $\det T = E_{n-2}$. So the original cofactor we wanted is $-E_{n-2}$, and the determinant of the original matrix is $E_{n-1} - E_{n-2}$.
(b) Using $E_n = E_{n-1} - E_{n-2}$, we get

$$\begin{aligned} E_3 &= E_2 - E_1 = 0 - 1 = -1 \\ E_4 &= E_3 - E_2 = -1 - 0 = -1 \\ E_5 &= E_4 - E_3 = -1 - (-1) = 0 \\ E_6 &= E_5 - E_4 = 0 - (-1) = 1 \\ E_7 &= E_6 - E_5 = 1 - 0 = 1 \\ E_8 &= E_7 - E_6 = 1 - 1 = 0 \end{aligned}$$

(c) The pattern is $1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \dots$, which repeats every 6 terms. So $E_{100} = E_{100-6\cdot 16} = E_4 = -1$.

(5) (a) When we reduce the block matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ to row-echelon form, its pivots are the pivots of A and the pivots of D, so this explains why the determinants match up (at least up to sign). To get the formula on the nose, we just notice that any row exchanges

that are done will multiply either $\det A$ or $\det D$ by -1 as well as the determinant of the whole matrix.

(b) Consider the case $B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and both A and D are the zero matrix. Then $\det A \det D = 0$ but the determinant of the block matrix is 1.

(c) Use the example from (b): $\det(AD - CB) = -1$.

(6) (a) The first three columns must be linearly dependent because their span is contained in the span of $\{(1,0,0,0,0),(0,1,0,0,0)\}$.

(b) The big formula says to take all column permutations $P = (a,b,c,d,e)$ and evaluate the sum $\sum \det(P) A_{1,a}A_{2,b}A_{3,c}A_{4,d}A_{5,e}$. If $c \leq 3$ or $d \leq 3$ or $e \leq 3$, then the term in the sum has to be 0 because those entries in the matrix are 0. But at least one of them has to be ≤ 3 because (a,b,c,d,e) is a permutation: if they weren't, they would take the values 4 or 5, so either the value 4 or the value 5 is repeated (and this violates the definition of permutation).

(7) (a) $\det \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = 12 - 2 = 10$.

(b) We need one of $v, w, v + w$ to be the origin to use determinants. Let's translate by $-v$, so the corners are $(0,0)$, $w - v = (-2,2)$ and $w = (1,4)$. Then the area is $|\frac{1}{2} \det \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix}| = |(-8-2)/2| = 5$.

(c) We need one of $v, w, w - v$ to be the origin. Let's translate by $-w$ so the corners are $v - w = (2,-2)$, $(0,0)$, and $-v = (-3,-2)$. Then the area is $|\frac{1}{2} \det \begin{pmatrix} 2 & -2 \\ -3 & -2 \end{pmatrix}| = |(-4-6)/2| = 5$.

(8) The length of the first column is $\sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$ (this is an identity from trigonometry) and the length of the second column is $\sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$. So J is the product $1 \cdot r = r$.

(9)

$$\begin{pmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

To get its determinant, let's expand cofactors along the bottom row:

$$\cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta) + \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta).$$

Do some distributing to simplify:

$$\rho^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\sin^2\theta + \cos^2\theta).$$

Now use the identity $\sin^2\theta + \cos^2\theta = 1$ to simplfy it to

$$\rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi.$$

Finally, distribute once more and use the identity again:

$$\rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi,$$

which is our determinant.