18.06 Problem Set 2 Solutions

1. Do problem 5 from section 2.6.

   Solution Doing the elimination process, with matrices:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} \rightarrow E_{31}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

   Thus in this case

$$E = E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

   With

$$L = E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

2. Do problem 16 from section 2.6.

   Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} c = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

   gives

$$c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

   Then

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

   gives

$$x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

   Those are the forward elimination and back substitution steps for

$$Ax = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

3. Do problem 21 from section 2.6.

Solution

(a) For the first matrix A. The elimination matrices E_{31}, E_{41} and E_{42} are all identity matrices. Thus $L = E^{-1}$ keeps the 3 lower zeros at the start of rows; the entries L_{31}, L_{41} and L_{42} are all zero But U may not have the upper zero where $A_{24} = 0$. For example for

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

It can be seen at the first nontrivial elimination step that the entry U_{24} is -1.

(b) For the second matrix B. The elimination matrix E_{41} is the identity matrix, thus $L = E^{-1}$ has entry $L_{41} = 0$. Also the first row is not changed during the elimination process, thus U_{14} remains 0. The other 0 entries however can get filled in, for example if

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The entry U_{23} becomes -1 in $E_{21}A$ (and it is not changed afterwards), and U_{32} becomes -1 also in $E_{21}A$, and since $A_{23} = 0$ it is not changed during further elimination.

4. Do problem 13 from section 2.7.

Solution A cyclic

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

or its transpose will have $P^3 = I$: $(1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$.

$$\hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$$

for the same P has $\hat{P}^4 = \hat{P} \neq I$.

5. Do problem 19 from section 2.7.

Solution

(a) $R^T AR$ is n by n, and it is symmetric:

$$(R^T A^T R)^T = R^T A^T (R^T)^T = R^T AR.$$

(b) The jth diagonal entry $(R^T R)_{jj}$ is (column j of R)·(column j of R)=(length squared of column j)≥ 0.

6. Do problem 34 from section 2.7.

Solution An elementary row operation matrix has the form

$$E = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

with inverse

$$E^{-1} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$$

We want

$$H = E^{-}1A = E = \begin{pmatrix} 1 & 2 \\ 4-x & 9-2x \end{pmatrix}$$

to be symmetric, that is $2 = H_{12} = H_{21} = 4 - x$, thus $x = 2$. So the factorisation is

$$EH = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \text{(elementery matrix)} \cdot \text{(symmetric matrix).}$$

7. Do problem 10 from section 3.1.

Solution

(a) this is subspace:

- for $\mathbf{v} = (b_1, b_2, b_3)$ with $b_1 = b_2$ and $\mathbf{w} = (c_1, c_2, c_3)$ with $c_1 = c_2$ the sum $\mathbf{v} + \mathbf{w} = b_1 + c_1, b_2 + c_2, b_3 + c_3$ is in the same set as $b_1 + c_1 = b_2 + c_2$.
- for an element $\mathbf{v} = (b_1, b_2, b_3)$ with $b_1 = b_2$. $c\mathbf{v} = (cb_1, cb_2, cb_3)$ and $cb_1 = cb_2$, thus it is in the same set.

(b) this is not a subspace, for example for $\mathbf{v} = (1, 0, 0)$ $-\mathbf{v} = (-1, 0, 0)$ is not in the set.

(c) this is not a subspace, for example the vectors $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, 0, 1)$ are in the set, but their sum $\mathbf{v} + \mathbf{w} = (2, 1, 1)$ is not.

(d) this is a subspace:

- for two vectors $\mathbf{v}_1 = \alpha_1\mathbf{v} + \beta_1\mathbf{w}$ and $\mathbf{v}_2 = \alpha_2\mathbf{v} + \beta_2\mathbf{w}$ the sum $\mathbf{v}_1 + \mathbf{v}_2 = \alpha_1\mathbf{v} + \beta_1\mathbf{w} + \alpha_2\mathbf{v} + \beta_2\mathbf{w} = (\alpha_1 + \alpha_2)\mathbf{v} + (\beta_1 + \beta_2)\mathbf{w})$ is still the linear combination of $\mathbf{v}$ and $\mathbf{w}$.
- for an element $\mathbf{v}_1 = \alpha_1\mathbf{v} + \beta_1\mathbf{w}$. $c\mathbf{v}_1 = c\alpha_1\mathbf{v} + c\beta_1\mathbf{w}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ thus it is in the same set.

(e) this is subspace:

- for $\mathbf{v} = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$ and $\mathbf{w} = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 0$ the sum $\mathbf{v} + \mathbf{w} = b_1 + c_1, b_2 + c_2, b_3 + c_3$ is in the same set as $b_1 + c_1 + b_2 + c_2 + b_3 + c_3 = 0$.
- for an element $\mathbf{v} = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$. $c\mathbf{v} = (cb_1, cb_2, cb_3)$ and $cb_1 + cb_2 + cb_3 = 0$, thus it is in the same set.

(f) this is not a subspace, for example $(1, 2, 3)$ is in the set, but $-1(1, 2, 3) = (-1, -2, -3)$ is not in the set.

8. Do problem 22 from section 3.1.

Solution The system

$$A\mathbf{x} = \mathbf{b}$$

has a solution if $\mathbf{b}$ is in the subspace spanned by the columns of A, thus

(a) Solution for every b;

(b) Solvable only if $b_3 = 0$;

(c) Solvable only if $b_3 = b_2$.

9. Do problem 24 from section 3.1.

Solution The column space of AB is contained in (possibly equal to) the column space of A. The example $B = 0$ and $A \neq 0$ is a case when $AB = 0$ has a smaller column space than A.