18.06 Fall 2010 Exam 1 solutions
1 (30 pts.)
Start with the 3 by 4 matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix}.$$

(a) (10 pts.) What are all the special solutions to $Ax = 0$, and describe the nullspace of A.

Solution: We can do row operations to get the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The free columns are 1, 3, and 4. We get each special solution by setting one of x_1, x_3, x_4 to 1 and the other two to 0. The answers are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

and the nullspace of A is the span of these three vectors.

(b) (10 pts.) What is the rank of A, and describe the column space of A.

Solution: A has 1 pivot, so it has rank 1. Each column is a multiple of the second column, so the column space is spanned by $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

(c) (5 pts.) Find all solutions to $Ax = \begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix}$.

Solution: A particular solution to this problem is $x_p = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}$. We can add any vector in the nullspace of A to get another solution, and this gives all solutions, so the general form is

$$\begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

where c_1, c_2, c_3 are some numbers.

(d) (5 pts.) Can A be written as $A = uv^T$ for some vectors u and v? If so what are these vectors, or if not, why not?

Solution: Yes, we know that a matrix can be written as uv^T if and only if it has rank 1 (or 0). Since the second column spans the column space, we take it to be u. Then the entries of v say which multiples the other columns are of u. So the answer is

$$u = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad v^T = (0, 1, 2, 3).$$

2 (30 pts.)
Consider the matrix

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where $ps = rq$ and $pr \neq 0$.

(a) (5 pts.) Describe simply and clearly the column space of A.

Solution: The first column is nonzero since $p \neq 0$ and $r \neq 0$. The second column is a multiple of the first column. To see this, multiply the first column by q/p and use the assumption that $ps = rq$. So the column space is spanned by $\begin{bmatrix} p \\ r \end{bmatrix}$.

(b) (10 pts.) Write as simply as possible the special solution(s) to $Ax = 0$, if any.

Solution: Subtract r/p times the first row from the second row to transform A into the matrix $\begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$, where again we used that $ps = rq$ to simplify. So column 2 is the only free column because $p \neq 0$. Setting $x_2 = 1$ gives the special solution $\begin{bmatrix} -q/p \\ 1 \end{bmatrix}$.

(c) (5 pts.) What are all the solutions to $Ax = 0$?

Solution: All multiples of the vector $\begin{bmatrix} -q/p \\ 1 \end{bmatrix}$.

(d) (10 pts.) Write A as simply as possible in row reduced echelon form.

Solution: We did one row operation in (b). To finish, divide the first column by p to get

$$\begin{bmatrix} 1 & q/p \\ 0 & 0 \end{bmatrix}.$$

3 (20 pts.)
(In the questions below, you can choose any n that works for an example, or prove that for all n, there is no example.)

(a) (10 pts.) Can you find independent vectors v, w, x and y in some space R^n and where $A = vw^T + xy^T$ is invertible? or prove that no such example exists?

Solution: No. Matrices of the form vw^T always have rank ≤ 1. The sum of two matrices of rank ≤ 1 has rank ≤ 2. So if A is invertible we need $n \leq 2$. But we also have the requirement that v, w, x, y are independent vectors, which forces $n \geq 4$. So we can't pick any n to make both inequalities happen.

(b) (10 pts.) Can you find vectors v, w, x and y that span some space R^n and where $A = vw^T + xy^T$ is invertible? or prove that no such example exists?

Solution: Yes. There are many possibilities. The easiest is to take $n = 1$. Then v, w, x, y are just numbers and A being invertible means it is nonzero. So we could take $v = w = 1$ and $x = y = 0$ for example.

4 (20 pts.)

Write an informal computer program to calculate xx^Tx, for any $n \times 1$ column vector x. The program should only use about $2n$ operations and no more than about $2n$ numbers in memory. You can write the program in MATLAB or your favorite language. It is not important that you remember exact syntax, but it is important that your operations are clear and unambiguous.

Solution: The naive way to do this is to first multiply xx^T and then multiply by x. But xx^T is a $n \times n$ matrix so doesn't meet the requirement of having roughly $2n$ numbers in memory (or the $2n$ operations requirement).

The point is that matrix multiplication is associative, so we can instead do $x(x^Tx)$, i.e., calculate x^Tx which is a single number, and then multiply x by this number. If the entries of x are $x_1, \dots, x_n$, then the final answer would be

$$\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \quad \text{where } c = x_1^2 + x_2^2 + \cdots + x_n^2.$$

So a program that does this should roughly look like this:

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c := 0;
for i from 1 to n do c := c + x[i]*x[i];
for i from 1 to n do answer[i] := x[i]*c;
return answer;
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