18.06 Problem Set 8 Solutions

Problem 1: Do problem 18 in section 6.4.

Solution

1. Suppose that $A = A^{\mathsf{T}}$ and $Ax = \lambda x$, $Ay = 0y$, and $\lambda \neq 0$. Then x is in the column space of A, and y is in the left nullspace of A since $\mathcal{N}(A) = \mathcal{N}(A^{\mathsf{T}})$. But $\mathcal{C}(A)$ and $\mathcal{N}(A^{\mathsf{T}})$ are orthogonal complements, so x and y are perpendicular.

2. If $Ay = \beta y$ with $\beta \neq \lambda$, then $(A - \beta I)x = (\lambda - \beta)x$ and $(A - \beta I)y = 0$. Since $\lambda - \beta \neq 0$ it follows that x is in the column space of $A - \beta I$ and y is in the nullspace of $A - \beta I$, and $(A - \beta I)^{\mathsf{T}} = A^{\mathsf{T}} - \beta I^{\mathsf{T}} = A - \beta I$. Therefore we can replace A with $A - \beta I$ in the argument of part 1 and it follows that x and y are perpendicular.

Problem 2: Do problem 21 in section 6.4.

Solution

(a) A matrix with real eigenvalues and eigenvectors is symmetric: **False**. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Then $\det(\lambda I - A) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The eigenvectors are $x_1 = (2, 1)^{\mathsf{T}}$ and $x_2 = (-1, 1)^{\mathsf{T}}$, so both the eigenvalues and eigenvectors are real but A is not symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric: **True**. If the matrix A has orthogonal eigenvectors $x_1, x_2, \ldots, x_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, we can define $s_i = \frac{x_i}{||x_i||}$ for all i; then $As_i = \lambda_i s_i$ for all i and the s_i are orthonormal. Then we can diagonalize A as

$$A = S\Lambda S^{-1}$$

where the ith column of S is s_i, and Λ is the diagonal matrix whose $(i.i)$th entry is λ_i. In particular S is an orthogonal matrix, so $S^{\mathsf{T}} = S^{-1}$ and $A = S\Lambda S^{\mathsf{T}}$. But now

$$A^{\mathsf{T}} = (S^{\mathsf{T}})^{\mathsf{T}} \Lambda^{\mathsf{T}} S^{\mathsf{T}} = S\Lambda S^{\mathsf{T}} = A$$

so A is symmetric.

(c) The inverse of a symmetric matrix is symmetric: **True**. If A is symmetric then it can be diagonalized by an orthogonal matrix Q, $A = Q\Lambda Q^{-1}$, and then $A^{-1} = Q\Lambda^{-1}Q^{-1} = Q\Lambda^{-1}Q^{\mathsf{T}}$. Since Λ^{-1} is still a diagonal matrix, it follows as in part (b) that $(A^{-1})^{\mathsf{T}} = Q\Lambda^{-1}Q^{\mathsf{T}} = A^{-1}$.

(d) The eigenvector matrix S of a symmetric matrix is symmetric: **False**. Example 1 in section 6.4 computes the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

to be $\lambda_1 = 0$, $x_1 = (2, -1)^{\mathsf{T}}$ and $\lambda_2 = 5$, $x_2 = (1, 2)^{\mathsf{T}}$. We can diagonalize A with eigenvector matrix

$$S = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

which is not symmetric.

Problem 3: Do problem 27 in section 6.4.
Solution The matrix

$$A + tB = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + t\begin{pmatrix} 8 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+8t & t \\ t & 2 \end{pmatrix}$$

has characteristic polynomial $\det(\lambda I - A) = \lambda^2 - (8t+3)\lambda + (2 + 16t - t^2)$, so its eigenvalues are

$$\begin{aligned} \lambda &= \frac{1}{2}\left(8t + 3 \pm \sqrt{(8t+3)^2 - 4(2 + 16t - t^2)}\right) \\ &= \frac{1}{2}\left(8t + 3 \pm \sqrt{68t^2 - 16t + 1}\right) \end{aligned}$$

and $||\lambda_1 - \lambda_2|| = \sqrt{68t^2 - 16t + 1} = \sqrt{(4(17t-2)^2 + 1)/17}$. This is minimized at $t = \frac{2}{17}$, so the minimum distance between the two eigenvalues is $||\lambda_1(t) - \lambda_2(t)|| = \sqrt{1/17} \approx 0.243$.

Problem 4: Do problem 7 in section 6.5.
Solution Using the fact that $R^{\mathsf{T}}R$ is positive definite if and only if the columns of R are independent, it is easy to check that $R^{\mathsf{T}}R$ is positive definite for

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$$

but not for

$$R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

since its columns satisfy the equation

$$3\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Problem 5: Do problem 12 in section 6.5.

Solution Since a 3×3 matrix is positive definite if and only if its three upper left determinants are positive, the matrix

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$$

is positive definite precisely when $\det(c) > 0$, $\det \begin{pmatrix} c & 1 \\ 1 & c \end{pmatrix} > 0$, and $\det(A) > 0$. The first two conditions are $c > 0$ and $c^2 - 1 > 0$, so we require $c > 1$. Using the big formula we compute

$$\begin{aligned} \det(A) &= c^3 + 1 + 1 - c - c - c \\ &= c^3 - 3c + 2 \\ &= (c-1)^3 + 3(c-1)^2, \end{aligned}$$

and so for $c > 1$ we see that $\det(A) > 0$ as well. Therefore A is positive definite whenever $c > 1$.

The three upper left determinants of

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

are 1, $\det \begin{pmatrix} 1 & 2 \\ 2 & d \end{pmatrix} = d - 4$, and $\det(B)$; the first two are positive for all $d > 4$. Using the big formula again, we find

$$\begin{aligned} \det(B) &= 5d + 24 + 24 - 16 - 20 - 9d \\ &= -4d + 12 \end{aligned}$$

which is positive for $d < 3$. But we can't satisfy both $d > 4$ and $d < 3$, so B is never positive definite.

Problem 6: Do problem 22 in section 6.5.

Solution The matrix $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 9$ with eigenvectors $x_1 = (-1, 1)^\mathsf{T}$ and $x_2 = (1, 1)^\mathsf{T}$, and both of these eigenvectors have length $\sqrt{2}$, so the eigenvector matrix $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ is orthogonal and $A = Q\Lambda Q^\mathsf{T}$ with $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$. The positive definite symmetric square root of A is then

$$\begin{aligned} R = Q\Lambda^{1/2}Q^\mathsf{T} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \end{aligned}$$

and we can check that

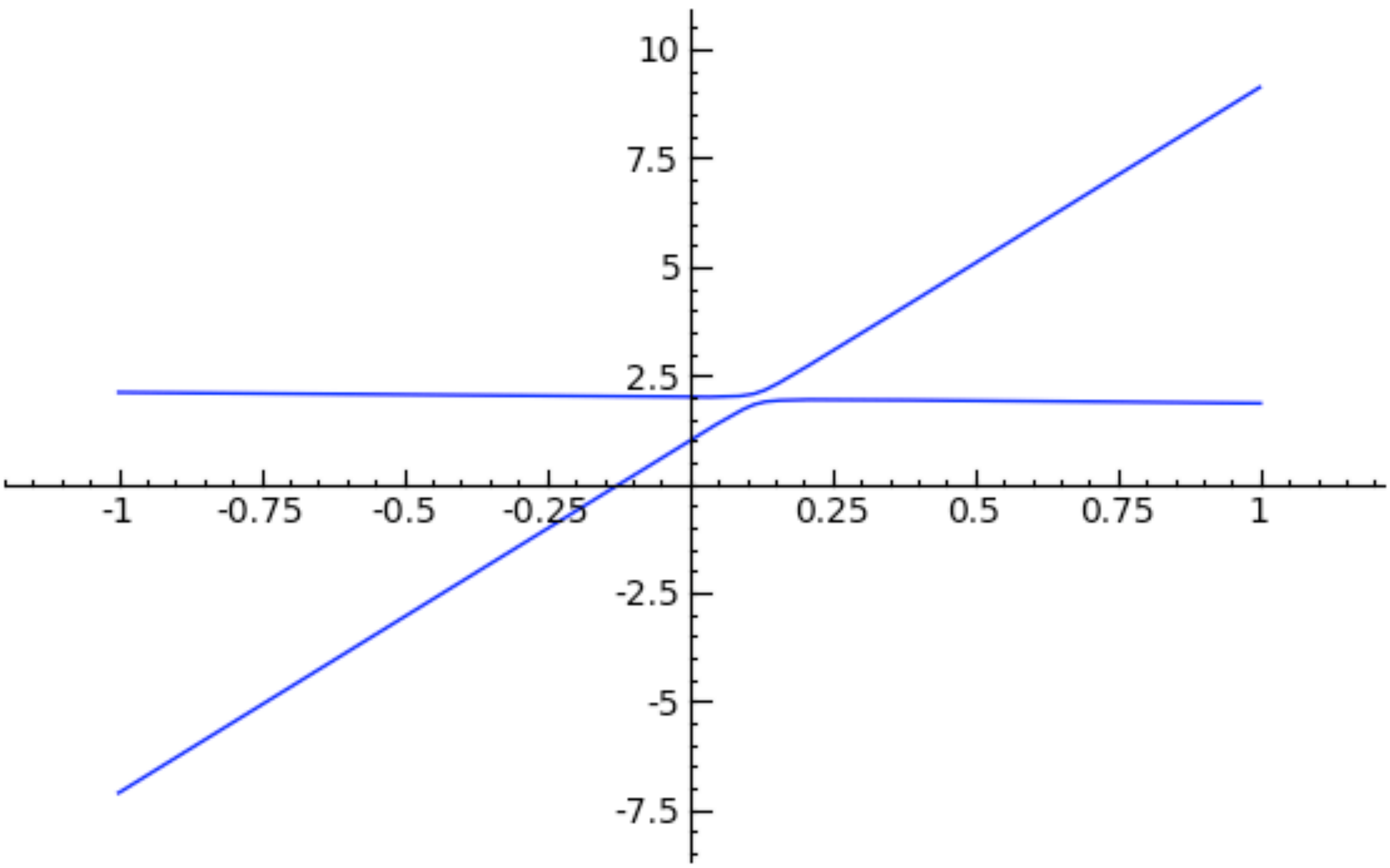
$$R^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = A.$$

The matrix $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 16$ with eigenvectors $x_1 = (-1, 1)^\mathsf{T}$ and $x_2 = (1, 1)^\mathsf{T}$, so $A = Q\Lambda Q^\mathsf{T}$ where Q is the same eigenvector matrix as before and $\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}$. Its square root is

$$\begin{aligned} R = Q\Lambda^{1/2}Q^\mathsf{T} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \end{aligned}$$

and indeed

$$R^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = A.$$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1