18.06 Problem Set 6 Solutions

Problem 1: Do problem 39 from section 5.3.

Solution Recall that $A^{-1} = C^T/det(A)$. If we know $det(A)$, then we get A^{-1}, hence find A. For the determinant, take determinants of both sides of the above equation. we have, $det(A^{-1}) = 1/det(A) = det(C^T)/det(A)^4$, hence $det(A) = det(C)^{1/3}$. We are done.

Problem 2: Do problems 6 from section 6.1.

Solution A is (lower) triangular, hence its eigenvalues are the entries on diagonal: 1 with multiplicity 2. Similarly B is (upper) triangular, hence its eigenvalue is 1 with multiplicity 2. The characteristic equations of AB and BA are both $\lambda^2 - 4\lambda + 1 = (\lambda - 2)^2 - 3 = 0$, hence their eigenvalues are $2 \pm \sqrt{3}$.

(a) The eigenvalues of AB are not the product of eigenvalues of A and B.

(b) AB and BA have the same characteristic equation, hence the same eigenvalues.

Problem 3: Problem 19 section 6.1.

Solution (a) B has 0 as its eigenvalue with multiplicity 1. Hence its null space has dimension 1, and the rank is 2.

(b) $|B^T B| = |B^T||B| = |B|^2 = 0$.

(c) Let $B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $B_1^T B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $B_2^T B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. We see that even though B_1 and B_2 both have eigenvalues 0,1 and 2, $B_1^T B_1$ and $B_2^T B_2$ have different eigenvalues. So the information is not enough.

(d) $B^2 + I$ has eigenvalues 1,2 and 5. Hence $(B^2 + I)^{-1}$ has eigenvalues 1,1/2 and 1/5.

Problem 4: Problem 9 section 6.2.

Solution $A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$.

(a) The characteristic equation is $\lambda^2 - 1/2\lambda - 1/2 = (\lambda - 1)(\lambda + 1/2) = 0$, hence eigenvalues are 1 and -1/2. For 1, the eigenvector is $(1, 1)^T$, and for -1/2, the eigenvector is $(1, -2)^T$.

(b) As $n \to \infty$, $\Lambda^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $A^n \to \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1} =$
$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$

(c)$A^{\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$. Hence Gibonacci numbers approach 2/3.

Problem 5: Do problem 11 in section 6.2.

Solution (a) True, since 0 is not an eigenvalue.

(b)&(c) False. We cannot tell if A is diagonalizable or not. For example, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ both have eigenvalues 2,2 and 5, but the first one is diagonalizable (already diagonal) and the second one is not diagonalizable.

Problem 6: Do problem 12 in section 6.2.

Solution (a) False. $A = \begin{pmatrix} 4 & 1 \\ -16 & 12 \end{pmatrix}$ is a counterexample.

(b) True. A is a 2×2 matrix, and if it has two distinct eigenvalues then there must be two eigenvectors.

(c) True.

Problem 7: Let Q be an n by n orthogonal matrix. Let A, B, and C be n by n matrices.

(a) Show that $\det(QAQ^T) = \det(A)$.

Solution $\det(QAQ^T) = \det(Q)\det(A)\det(Q^T) = \det(A)\det(QQ^T) = \det(A)$.

(b) The trace of A is the sum of the diagonal entries. $\text{tr}A = \Sigma_{i=1}^n a_{ii}$. Show that $\text{tr}(BC) = \text{tr}(CB)$.

Solution $\text{tr}AB = \Sigma_{i=1}^n \Sigma_{j=1}^n a_{ij} b_{ji} = \Sigma_{j=1}^n \Sigma_{i=1}^n b_{ji} a_{ij} = \text{tr}BA$.

(c) Use the result of part (b) to show that $\text{tr}(QAQ^T) = \text{tr}(A)$.

Solution $\text{tr}(QAQ^T) = \text{tr}(AQ^TQ) = \text{tr}(A)$.

(d) Consider the matrix $A - \lambda I$. Use the big determinant formula to show that $\det(A - \lambda I)$ is a polynomial of degree n.

Solution In the big formula, determinant is expressed as a sum of product of entries from each column. Since each entry of $A - \lambda I$ is a polynomial of λ, its determinant is also a polynomial of λ. Moreover, it has the only one combination of entries that makes the largest degree polynomial: the product of diagonal entries. It has degree n, hence the determinant is a polynomial of degree n.

(e) So now we have

$$\det(A - \lambda I) = \Sigma_{i=0}^{n} c_i \lambda^i,$$

where c_i just denotes the coefficient of the term λ^i in this polynomial. In the case that the dimension of A is 2 by 2, identify the coefficients of this polynomial in terms of trace and determinant.

Solution When $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (\text{tr}A)\lambda + \det A$.

(d) Show that each coefficient c_i is invariant in the sense that, given orthogonal matrix Q:

$$\det(QAQ^T - \lambda I) = \det(A - \lambda I).$$

Solution $\det(QAQ^T - \lambda I) = \det(Q(A - \lambda I)Q^T) = \det(A - \lambda I)$