18.06 Problem Set 4 Solutions

Problem 1: Do problem 13 from section 3.6.

$\boxed{\text{Solution}}$

(a) If $m = n$ then the row space of A equals the column space.

FALSE. Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Here, $m = n = 2$ but the row space of A contains multiples of $(1, 2)$ while the column space of A contains multiples of $(1, 3)$.

(b) The matrices A and $-A$ share the same four subspaces.

TRUE. The nullspaces are identical because $A\mathbf{x} = \mathbf{0} \iff (-A)\mathbf{x} = \mathbf{0}$. The column spaces are identical because any vector v that can be expressed as $\mathbf{v} = A\mathbf{x}$ for some $\mathbf{x}$ can also be expressed as $\mathbf{v} = (-A)(-\mathbf{x})$. A similar reasoning holds for the two remaining subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

FALSE. Any invertible 2x2 matrix will have $\mathbf{R}^2$ as its column space and row space and the zero vector as its (left and right) nullspace. However, it is easy to produce two invertible 2x2 matrices that are not multiples of each other: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 2: Do problem 25 from section 3.6.

$\boxed{\text{Solution}}$

(a) A and A^{T} have the same number of pivots.

TRUE. The number of pivots of A is its column rank, r. We know that the column rank of A equals the row rank of A, which is the column rank of A^{T}. Hence, A^{T} must have the same number of pivots as A.

(b) A and A^{T} have the same left nullspace.

FALSE. Counterexample: Take any a 1x2 matrix, such as $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$. The left nullspace of A contains vectors in $\mathbf{R}$ while the left nullspace of A^{T}, which is the right nullspace of A, contains vectors in $\mathbf{R}^2$, so they cannot be the same.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

FALSE. Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Here, the row space and the column space are both equal to all of $\mathbf{R}^2$ (since A is invertible), but $A \neq A^{\mathrm{T}}$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

TRUE. The row space of A equals the column space of A^{T}, which for this particular A equals the column space of $-A$. Since A and $-A$ have the same fundamental subspaces by part (b) of the previous question, we conclude that the row space of A equals the column space of A.

Problem 3: Do problems 1 and 2 in section 8.2. Please note that these problems correspond to the triangular graph.

$\boxed{\text{Solution}}$

The incidence matrix for the triangle graph is $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

In order to have a zero potential difference across every edge, the potentials must be equal. Hence, the nullspace of A contains multiples of the vector (1,1,1). Since the vector (1,0,0) is not perpendicular to the vector (1,1,1) in the nullspace, it cannot be in the rowspace.

The transpose of the incidence matrix is $A^{\mathrm{T}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

The vector $\mathbf{y} = (1, -1, 1)$ is in its nullspace. This vector corresponds to a current of 1 going around the loop formed by edges 1, 3 and the reverse of edge 2 (i.e. current is flowing clockwise, in the direction of edges 1 and 3 but in the opposite direction of edge 2).

Please note that problems 4 and 5 both correspond to the square graph.

Problem 4: Do problem 8 in section 8.2.

Solution

The incidence matrix for the square graph is $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

One solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = (1, 1, 1, 1)$. In order to solve $A^{\mathrm{T}}\mathbf{y} = \mathbf{0}$, we need to identify the closed loops in the graph. Here, there are two such loops, one formed by edges 1,3, and the reverse of edge 2, and one formed by edges 3, 5 and the reverse of edge 4. Hence, the vectors $\mathbf{y_1} = (1, -1, 1, 0, 0)$ and $\mathbf{y_2} = (0, 0, 1, -1, 1)$ both solve $A^{\mathrm{T}}\mathbf{y} = \mathbf{0}$.

Problem 5: Do problem 13 in section 8.2.

Solution

By computing $A^T CA$, we get

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix}$$

We can solve $A^T CA\mathbf{x} = \mathbf{f}$ by grounding node 4, which gives the system of equations

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 8 & -3 \\ -2 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this system gives $x_1 = \frac{5}{12}, x_2 = \frac{1}{6}, x_3 = \frac{1}{6}$ (and $x_4 = 0$ because we decided to ground node 4). We can now compute the currents as

$$\mathbf{y} = -CA\mathbf{x} = - \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{12} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Problem 6: Do problem 21 in section 4.1.

Solution

If $\mathbf{S}$ is spanned by (1,2,2,3) and (1,3,3,2), then $\mathbf{S}^\perp$ contains all the vectors orthogonal to (1,2,2,3) and (1,3,3,2). To find a basis for $\mathbf{S}^\perp$ we need to solve $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$. Reducing A to row-echelon form gives $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}$. By setting the pivot variables to zero in turn, we conclude that the nullspace is spanned by (0,1,-1,0) and (-5,0,1,1).

Problem 7: Do problem 29 in section 4.1.

Solution

The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ contains $\mathbf{v}$ in both its row space and column space.

The matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{bmatrix}$ contains $\mathbf{v}$ in both its nullspace and column space.

$\mathbf{v}$ cannot be in both the row space and the nullspace of some A, or both in its column space and its left nullspace, since otherwise we would have $\mathbf{v}^T\mathbf{v} = 0 \implies \mathbf{v} = 0$.

Problem 8: Do problem 32 in section 4.1.

Solution

(a) We know that the row space $C(A^{\mathrm{T}})$ needs to be orthogonal to the nullspace $N(A)$, and that the column space $C(A)$ needs to be orthogonal to the left nullspace $N(A^{\mathrm{T}})$. Since the matrix A is 2x2 and all the fundamental subspaces are 1-dimensional, this translates into two conditions: $\mathbf{r}^{\mathrm{T}}\mathbf{n} = 0$ and $\mathbf{c}^{\mathrm{T}}\mathbf{l} = 0$.

(b) Since we have bases for the row and column space of A, we can simply take $A = \mathbf{c}\mathbf{r}^{\mathrm{T}}$ as our matrix. Since $\mathbf{c}$ and $\mathbf{r}$ are nonzero, this gives us the correct row and column spaces, and since the conditions in (a) hold, the nullspaces are also correct. For instance, if $\mathbf{c} = (1,2)$ and $\mathbf{r} = (2,1)$, we get $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.