18.06 Professor Postnikov Quiz 2 October 26, 2009

SOLUTIONS ∇
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Your PRINTED name is: ____________________

Please circle your recitation:

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Grading

1

2

3

Total:

Problem 1. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}$.

(a) Find an orthogonal basis of the column space of the matrix A.

(b) Find a non-zero vector $\mathbf{v}$ which is orthogonal to the column space of A.

(c) Does this vector $\mathbf{v}$ belong to one of the four fundamental subspaces of A? Which subspace? Explain why.

(d) Find a 3 by 2 matrix Q with $Q^TQ = I$ such that Q has the same column space as the matrix A.

(a) Use Gram-Schmidt on the colums of $A = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}$.

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$q_2 = \frac{a_2 - (a_2^T \cdot q_1) q_1}{\|a_2 - (a_2^T q_1) q_1\|}, \quad a_2^T \cdot q_1 = \frac{9}{\sqrt{3}} \quad \text{so} \quad q_2 = \frac{\begin{pmatrix}1\\3\\5\end{pmatrix} - \underbrace{\frac{9}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}}_{3}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\| \quad '' \quad \|}$$

$$= \begin{pmatrix}-2\\0\\2\end{pmatrix} \cdot \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\boxed{\text{An orthogonal basis is given by } \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} \right\}.}$$

(b) Must find v s.t. $a_1^T v = 0$ and $a_2^T v = 0$ or similarly

$q_1^T v = 0$ and $q_2^T v = 0$ (since q_1 and q_2 are a basis for $C(A)$!)

This is the same as solving:

$$\begin{pmatrix} - & q_1^T & - \\ - & q_2^T & - \end{pmatrix} v = 0 \quad \text{or} \quad \begin{pmatrix} \frac{1}{\sqrt{3}}(1 & 1 & 1) \\ \frac{1}{\sqrt{2}}(1 & 0 & -1) \end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{matrix}0\\0\\0\end{matrix}$$

$$\Rightarrow \begin{array}{l} v_1 - v_3 = 0 \\ v_1 + v_2 + v_3 = 0 \end{array} \quad \text{So} \quad \begin{array}{l} v_1 = v_3 \\ v_2 = -2v_1 \end{array} \quad \text{and} \quad \boxed{v = \begin{pmatrix}1\\-2\\1\end{pmatrix}}$$

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(c) $v \in N(A^T)$. Why? Let Q be the matrix with columns given by q_1 and q_2 from part (a).

Then $C(A) = C(Q) \Longrightarrow N(A^T) = N(Q^T)$

And from part (b), we saw that v was a solution to $Q^T v = 0$. Since $C(Q)$ is 2-dim $N(Q^T)$ is 1-dimensional, so is actually spanned by v.

(d) Take Q to be Q from part (c).

$$Q^T Q = \begin{pmatrix} - & q_1^T & - \\ - & q_2^T & - \end{pmatrix} \begin{pmatrix} | & | \\ q_1 & q_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \overset{1}{\overset{\shortparallel}{q_1^T q_1}} & \overset{0}{\overset{\shortparallel}{q_1^T q_2}} \\ \underset{0}{\underset{\shortparallel}{q_2^T q_1}} & \underset{1}{\underset{\shortparallel}{q_2^T q_2}} \end{pmatrix} = I.$$

Problem 2. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}$, and let $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$.

(a) What is the projection of $\mathbf{b}$ onto the column space of A?

(b) Give an orthogonal basis for each of the four fundamental subspaces of A.

(c) Use least squares approximation to solve $A\mathbf{x} = \mathbf{b}$.

(a) Want to find $p = A\hat{x}$ (i.e. in $C(A)$) closest to b.
Or equivalently $A^T(b - A\hat{x}) = 0 \Rightarrow A^TA\hat{x} = A^Tb$
Since columns of A are independent $\hat{x} = (A^TA)^{-1}A^Tb$
So the projection $A\hat{x}$ is given by:

$$\underset{\displaystyle A\hat{x}}{\overset{\shortparallel}{p}} = A(A^TA)^{-1}A^Tb:$$

$$A^TA = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\text{So } (A^TA)^{-1} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}$$

$$p = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}\overbrace{\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}}^{\begin{pmatrix} 0 \\ -6 \end{pmatrix}}$$

$$\begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix}\begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

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(b) From part (a): $A^T A = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$ shows that the columns of A are already orthogonal. They are clearly independant as A has 2 pivots, so

$\boxed{C(A) \text{ has basis: } \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}}$

Since $\dim C(A) = \dim C(A^T) = 2$, the rowspace is all of $\mathbb{R}^2$, so we can take any basis

or $\boxed{C(A^T) \text{ has a basis } \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}$

(These are clearly orthogonal.)

$\boxed{N(A) = \vec{0} \text{ since } A \text{ has full rank.}}$

$N(A^T)$ is spanned by a vector v s.t. $A^T v = 0$, so

$\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right\}$ works.

(c) Want to solve $A^T A \hat{x} = A^T b$ or $\hat{x} = (A^T A)^{-1} A^T b$.

From part (a) $\hat{x} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}$

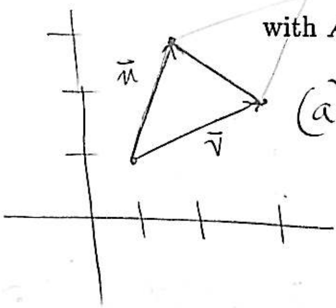
Problem 3.

(a) Find the area of the triangle on the plane $\mathbb{R}^2$ with the vertices $(1,1)$, $(2,3)$, $(3,2)$.

(b) Calculate the determinant of the 4 by 4 matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(c) Find the inverse of the matrix A from part (b). Check your answer by multiplying it with A.



(a) Take $\vec{u} = (2,3) - (1,1) = (1,2)$

$\vec{v} = (3,2) - (1,1) = (2,1)$.

Then the Area of triangle $= \frac{1}{2}$ area of parallelogram.

So $\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} \vec{u} & \vec{v} \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right| = \frac{1}{2}|1-4| = \boxed{\frac{3}{2}}$

(b) $\det A = 1 \cdot \det\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} - (-1)\det\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ $\to 0$

$= 1 - (1+1) = \boxed{-1}$

row 2 and 3 are dependant, so det of this matrix $= 0$

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There is more than one way to do this, but the fewest operations are required by the following:

(c):
$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

row 2 $\rightarrow$ row 2 + row 1
row 3 $\rightarrow$ row 3 + row 4

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

row 1 $\rightarrow$ row 1 - row 3
row 4 $\rightarrow$ row 4 - row 2

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

multiply row 2 + 3 by (-1) and switch them.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

Check $A^{-1}A = I$:
$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$