Problem 1. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Find the factorization $A = LU$.

(b) Find the inverse of A.

(c) For which values of c is the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & c \end{pmatrix}$ invertible?

Solution

(a) We row reduce A by subtracting row 1 from row 2 (E_{12}) and then add row 2 to row 3 (E_{23}) to find the upper triangular matrix

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since we can reverse this process and subtract row 2 from row 3 in U, followed by adding row 1 to row 2 to obtain A, we see that the lower triangular matrix is the product:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Hence we find

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Note that $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$. We explicitly compute U^{-1} and find:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Similary, we compute

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} A^{-1} &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 3 & 1 \\ 3 & -3 & -2 \\ -1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

(c) We row reduce to find :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & c \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & c-2 \end{pmatrix}.$$

Note that A is invertible if and only if it has 3 nonzero pivots. Thus A is invertible when $c \neq 2$.

Problem 2. Which of the following are subspaces? Explain why.

(a) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{x}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$.

(b) All vectors $(x, y)^T$ in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$.

(c) All vectors $(x, y)^T$ in $\mathbb{R}^2$ such that $x + y = 2$.

(d) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ which are in the column space AND in the nullspace of the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

(e) All vectors $\mathbf{x}$ in $\mathbb{R}^3$ which are in the column space OR in the nullspace (or in both) of the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

Solution

(a) Yes. This equation describes the left nullspace of the matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Since the left nullspace is a vector space, we are done.

(b) No. Consider the vectors $(1, 1)^T$ and $(1, -1)^T$, both of which satisfy this equation. The sum $(1, 1)^T + (1, -1)^T = (2, 0)^T$ does not satisfy the equation since $2^2 - 0 = 4$.

(c) No. This set does not contain $(0, 0)^T$.

(d) Yes. Note that this column space of this matrix is the span of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The nullspace is spanned by the vectors: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Since the sum of the two nullspace basis vectors is the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the intersection is the column space, which is a vector space.

(e)Yes. We have already seen in part (d) that the column space is a subspace of the nullspace. Thus the vectors that are in the column space or the nullspace are just the columns in the nullspace, which is a vector space.

Problem 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find the complete solution of the equation $A\mathbf{x} = \mathbf{0}$.

(b) Find the complete solution of the equation $A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(c) Find all vectors $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(d) Find a matrix B such that $N(A) = C(B)$.

(e) Find bases of the four fundamental subspaces for the matrix A.

Solution

In preparation for the next problems, lets first row reduce this matrix with an arbitrary vector augmented.

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ -1 & -2 & 0 & 0 & -1 & b_2 \\ 1 & 2 & 0 & 0 & 1 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_1 + b_2 \\ 0 & 0 & -1 & -2 & -1 & b_3 - b_1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 & 2 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & 0 & b_2 + b_3 \end{pmatrix}$$

(a) We are finding the nullspace, or the solution when $b_1 = b_2 = b_3 = 0$. Since this matrix has 2 pivots and 3 free columns, so a general solution is just any linear combination of the nullspace basis vectors:

$$c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) To find a general solution, we set $b_1 = 2$, $b_2 = 1$ and $b_3 = -1$ in our augmented matrix row reduction performed above. We find a particular solution to

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix},$$

which is a vector with zeros in the free variables, so we solve directly and find our particular solution:

$$\mathbf{x}_p = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

Thus a general solution is just this particular solution plus the general solution for a nullspace vector given in part (a):

$$\begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Finding all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution is asking for condition on $\mathbf{b}$ so that it is in the column space. From our original computation, we see that we must have

$b_2 + b_3 = 0$, so an arbitrary vector in the column space looks like $\begin{pmatrix} b_1 \\ b_2 \\ -b_2 \end{pmatrix}$, where b_1 and b_2 are any real numbers. In otherwords, the column space is spanned by vectors of the form:

$$b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(d) To find a matrix B such that $N(A) = C(B)$, we can simply take a matrix whose column vectors are a basis for the nullspace of A. In otherwords, the matrix:

$$B = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(e) Observe that the three vectors in part (a) form a basis for the nullspace of A, the two vectors in part (c) form a basis for the column space of (a). Thus all that is left is to find basis vectors for the row space, which we can take to be the two independent row vectors corresponding to the pivot rows of A: $\begin{pmatrix} 1 & 2 & 1 & 2 & 2 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \end{pmatrix}^T$. A basis for the left null space is given by the vector that is in the nullspace of the matrix whose rows are the basis vectors for the column space of A:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Thus a basis for the left nullspace is given by the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Problem 4. Let A be an m by n matrix. Let B be an n by m matrix. Suppose that $AB = I_m$ is the m by m identity matrix.

1. Let $r = \text{rank}(A)$ denote the rank of the matrix A. Choose one answer and be sure to justify it.

   (a) $r \geq m$

   (b) $r \leq m$

   (c) $r = m$

   (d) $r > n$

2. Is $m \leq n$ or is $n \leq m$? Why?

Solution

1. The rank of A is equal to the dimension of the column space $C(A)$. Now the column space of A is the subspace of $\mathbb{R}^m$ that can be written as $A\mathbf{x}$, where $\mathbf{x}$ is a vector in $\mathbb{R}^n$. I claim that $C(A) = \mathbb{R}^m$. This follows because the column space AB is the column space of the identity matrix is all of $\mathbb{R}^m$. So in particular, any vector $\mathbf{v}$ in $\mathbb{R}^m$ can be written as $AB\mathbf{v} = I_m\mathbf{v} = \mathbf{v}$. Thus any vector $\mathbf{v}$ in $\mathbb{R}^m$ is in the column space of A because it can be written as $\mathbf{v} = A\mathbf{x}$ by simply setting $\mathbf{x} = B\mathbf{v}$. Therefore we have seen that the dimension of the column space is m, and thus the answer is (c).

2. We know that the rank of A must be less than or equal to the smallest dimension of A. Since $r = m$, it must be the case that $m \leq n$.