SOLUTIONS TO PSET 9

Problem 1. (5 points each)

1. $B = M^{-1}AM$ and $C = N^{-1}BN$ imply $C = N^{-1}M^{-1}AMN = (MN)^{-1}A(MN)$. If B is similar to A and C is similar to B, then A is similar to C.

2. $F^{-1}AF = C = G^{-1}BG$, so $B = GF^{-1}AFG^{-1} = (FG^{-1})^{-1}A(FG^{-1})$. If C is similar to A and also to B, then A and B are similar.

Problem 2. Let M be a 4×4 matrix. Then $JM = \begin{pmatrix} m_{21} & m_{22} & m_{32} & m_{42} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ while $MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$. If $JM = MK$, then we conclude from comparing these two matrices that $m_{11} = m_{22} = 0$, and $m_{21} = 0$, and $m_{31} = m_{42} = 0$, and $m_{41} = 0$. Thus $det(M) = 0$ and M is not invertible as required.

Problem 3. (2.5 points each)

a) True. If $A = MBM^{-1}$ with B invertible, then $det(A) = det(M)det(B)det(M^{-1}) = det(B) \neq 0$.

b) False. $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 10 & -3 \\ 15 & -5 \end{pmatrix}$.

c) False. We know (problem 13) that A and A^t are similar. So just choose a nonzero skew-symmetric matrix.

d) True. If A is similar to $A+I$, then $tr(A) = tr(A+I) = tr(a)+n$, which is impossible.

Problem 4. We have that $\{\mathbf{v}_i\}$ and $\{\mathbf{u}_i\}$ are orthonormal bases in $\mathbb{R}^n$. We want A such that $A\mathbf{v}_i = \mathbf{u}_i$. If we let V be the matrix whose columns are the $\mathbf{v}_i$, and U the matrix whose columns are the $\mathbf{u}_i$, then what we are asking for is $AV = U$, or, equivalently, $A = UV^t$.

Problem 5. Here, we suppose that the $m\times n$ matrix A has orthogonal columns, labelled $\{\mathbf{w}_i\}$, with lengths $\{\sigma_i\}$. This tells us that $A^tA = \Lambda$, where Λ is the diagonal matrix with eigenvalues σ_i^2. Thus $V = I$. So the SVD reads $A = U\Sigma$. So we can let Σ be the $m\times n$ matrix whose first n diagonal elements are the σ_i and all of whose other elements are 0 (note that $m \geq n$ because the columns of A are orthogonal, hence independent), and we can let U be the orthogonal $m\times m$ matrix whose first n columns are $(1/\sigma_i)\mathbf{w}_i$, and the rest of whose columns form an orthonormal basis for the left nullspace of A.

Problem 6. (5 points each)

1. We have $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $T(M) = AM$. Then $T(M_1+M_2) = A(M_1+M_2) = AM_1+AM_2 = T(M_1)+T(M_2)$. $T(\lambda M) = A(\lambda M) = \lambda(AM) = \lambda T(M)$; so T is linear.

2. Suppose $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. Then $det(A) = -1$, so A is invertible. Now, $T(M) = AM = 0$ implies $0 = A^{-1}(AM) = M$. Further, given B, $T(A^{-1}B) = A(A^{-1}B) = B$.

Problem 7. (2 points for 15, 2 each for 17)

1. Now put $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Then $det(A) = 0$, so A is not invertible. Thus $T(M) = AM = I$ is impossible. Further, $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} = 0$.

2. a) True. $T^2(A) = TT(A) = T(A^t) = (A^t)^t = A$.

b) True. T is invertible (it is its own inverse, by part a), so $Ker(T) = 0$.

c) True. For any B, $B = T(B^t)$.

d) False. This is just the skew-symmetry condition.

Problem 8. Clearly we have $B = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ as $S(1) = 0$, $S(x) = 0$, $S(x^2) = 2$, $S(x^3) = 6x$.

Problem 9. (5 points each)

1.The matrix for T is $\begin{pmatrix} Tv_1 & Tv_2 & Tv_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Further, $T(v_1 - v_2) = w_1 + w_2 + w_3 - (w_2 + w_3) = w_1$.

2. Well, $T^{-1}(w_3) = v_3$ clearly. Next, $T^{-1}(w_2 + w_3) = v_2$, so $T^{-1}(w_2) = v_2 - v_3$. Finally, $v_1 = T^{-1}(w_1 + w_2 + w_3) = T^{-1}(w_1) + v_2 - v_3 + v_3 = T^{-1}(w_1) + v_2$, so $T^{-1}(w_1) = v_1 - v_2$. Thus $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. $T\mathbf{v} = 0$ only happens when $\mathbf{v} = 0$, because T is invertible.

Problem 10. We have $A = QR$. Now, any invertible matrix B can be interpreted as the c.o.b. matrix from the basis which constits of columns of B to the standard basis (this is his defintion of c.o.b. matrix in the text). Thus, A is the c.o.b. matrix from the basis $\{a_1, a_2, a_3\}$ to the standard basis and Q is the c.o.b. from the basis $\{q_1, q_2, q_3\}$ to the standard basis. So Q^{-1} is the c.o.b. matrix from the standard basis to the basis $\{q_1, q_2, q_3\}$. So $R = Q^{-1}a$ is the c.o.b. matrix from the basis $\{a_1, a_2, a_3\}$ to the basis $\{q_1, q_2, q_3\}$.