SOLUTIONS TO PSET 6

Problem 1. (5 points each) a) $1 = det(I) = det(Q^tQ) = det(Q^t)det(Q) = (det(Q))^2$, so $det(Q) = \pm 1$.

b) Suppose that $|det(Q)| > 1$. Then we have that, for each m, $|det(Q^m)| = |det(Q)|^m$, which goes to ∞ as $m \to \infty$. But Q^m is an orthogonal matrix for each m. So we have to show that $|det(U)|$ is uniformly bounded as U ranges over all orthonormal matrices. But consider the big sum: $|det(U)| \le \sum_\sigma |u_{1\sigma(1)}u_{2\sigma(2)}...u_{n\sigma(n)}|$. Since U is orthogonal, each column has length 1, and so we have $|u_{ij}| \le 1$ for all i, j. But then we can estimate $|det(U)| \le \sum_\sigma 1 = n!$; so we have obtained a uniform bound as required. If instead we had started with $|det(Q)| < 1$, then we simply replace Q with Q^{-1}(which is also orthogonal) and repeat the argument.

Problem 2. (5 points each)

1. $\begin{pmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{pmatrix} \to \begin{pmatrix} 101 & 100 & 200 \\ 102 & 100 & 200 \\ 103 & 100 & 200 \end{pmatrix}$, so $det = 0$.

2. $\begin{pmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{pmatrix} \to \begin{pmatrix} -t^2+1 & 0 & 0 \\ t & 1 & t \\ 0 & 0 & -t^2+1 \end{pmatrix} \to \begin{pmatrix} -t^2+1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & -t^2+1 \end{pmatrix}$ so $det = (1-t^2)^2$.

Problem 3. (2.5 points each)

a) True; $det(A) = 0$ implies $det(AB) = det(A)det(B) = 0$.

b) False, just consider a matrix where you need one row exchange: $det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = -det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1$.

c) False, let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

d)True, both $det(AB)$ and $det(BA)$ are equal to $det(A)det(B)$.

Problem 4. (5 points each)

a) If you look at the first three columns, you see three vectors of the form $\begin{pmatrix} x \\ x \\ 0 \\ 0 \\ 0 \end{pmatrix}$. But vectors of this form span a two-dimensional subspace. Since three vectors in a two dimensional subspace are linearly dependent, the determinant must be zero.

b) Take any term in the big sum. It is a product of five terms, three of which come from the first three columns. As these three must come from three distinct rows, one of them must be zero.

Problem 5. (5 points each)

1. For $A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$, se have that $M_{11} = 6$, $M_{12} = 3$, $M_{21} = 1$, $M_{22} = 2$, so $C = \begin{pmatrix} 6 & -3 \\ -1 & 2 \end{pmatrix}$.

For $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{pmatrix}$, we have that $C = \begin{pmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{pmatrix}$ and $det(B) = 7 * -3 = -21$ (computing using the third row).

2. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, so $C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $AC^t = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ as required.

Problem 6. (4,3, and 3 points)

a) For this, just look at the big sum. Each term involves one element of each column. We need to select elements from the first two columns; only those selections with all elements from A can be nonzero. Then we need to select from the last two rows, but since one element from each column of A has been chosen, we can only select elements from D. Thus the determinant has four nonzero terms, it is easily verified that the product $(a_{11}a_{22} - a_{12}a_{21})(d_{11}d_{22} - d_{12}d_{21})$ consists of exactly the required terms.

b)The matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$ has orthogonal columns (in fact, $1/2$ times it is orthonormal), so it is invertible. However, $|A||D| - |B||C| = 1(-1) - (1)(-1) = 0$.

c) The matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ is singular; but the expression $det(AD - CB) = -1$.

Problem 7. (5 points each)

a) we get $x_1 = det(B_1)/det(A)$, Cramer's rule.

b) The middle equality is $det\begin{pmatrix} x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = det\begin{pmatrix} x_1\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} + det\begin{pmatrix} x_2\mathbf{a}_2 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} + det\begin{pmatrix} x_3\mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = x_1 det(A)$.

Problem 8. (5 points each)

a) we know that L^{-1} is given by $(1/det(L))C^t$, where C is the cofactor matrix. To see that C^t is lower triangular, we should show that C is upper triangular. But, the cofactors coming from b, d, and e are all determinants of 2×2 matrices with a row (or column) of zero's; hence are equal 0.

b) this follows by examining the cofactors coming from "mirror" terms in the matrix.

Problem 9. (5 points each)

1. The lengths of the two columns are 1 and r. Thus $J = r$.

2. The inverse of $\begin{pmatrix} cos(\theta) & -rsin(\theta) \\ sin(\theta) & rcos(\theta) \end{pmatrix}$ is $\begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta)/r & cos(\theta)/r \end{pmatrix}$, whose determinant is $1/r$. The chain rule he states here comes down to $cos^2(\theta) + sin^2(\theta) = 1$.