SOLUTIONS TO PROBLEM SET 1

Problem 1. (5+5 points). a) $(v_1w_1+v_2w_2)^2 = v_1^2w_1^2+2v_1w_1v_2w_2+v_2^2w_2^2$, while $(v_1^2+v_2^2)(w_1^2+w_2^2) = v_1^2w_1^2+v_1^2w_2^2+v_2^2w_1^2+v_2^2w_2^2$.

b) The difference $RHS-LHS = v_1^2w_2^2+v_2^2w_1^2-2v_1w_2v_2w_1 = (v_1w_2-v_2w_1)^2 \geq 0$.

Problem 2. (10 points). We suppose $||\mathbf{v}|| = 5$ and $||\mathbf{w}|| = 3$. We know that $\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||||\mathbf{w}||cos(\theta)$, so the minimum is when $\theta = \pi$ (and the dot product is -15), and the maximum is when $\theta = 0$ (and the dot product is 15). For the second part, $||\mathbf{v}-\mathbf{w}||$ is the length of the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$, and we have that $||\mathbf{v}-\mathbf{w}||^2 = (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}-2\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||^2+||\mathbf{w}||^2-2\mathbf{v}\cdot\mathbf{w} = 34-2\mathbf{v}\cdot\mathbf{w}$. So this function is maximized when $\mathbf{v}\cdot\mathbf{w}$ is minimized, and vice versa. So the max of $||\mathbf{v}-\mathbf{w}||$ occurs at $\sqrt{64} = 8$ and the minimum at $\sqrt{4} = 2$. We note that the maximum occurs when $\mathbf{v}$ and $\mathbf{w}$ face in opposite directions, and the minimum when they line up (the real way to see this is just to draw a picture in the plane).

Problem 3. (10 points). We are given two planes $x+y+3z = 6$ and $x-y+z = 4$. We to find a point lying on both with $z = 2$; so, we need to solve simultaniously $x+y+6 = 6$ and $x-y+2 = 4$, i.e., $x+y = 0$ and $x-y = 2$. Solve by adding the equations to obtain that $(x = 1, y = -1)$ provides the solution, so we obtain the point $(1,-1,2)$. If instead we have $z = 0$, the equations become $x+y = 6$ and $x-y = 4$, and obtain the solution $(x = 5, y = 1)$; giving the point $(5,1,0)$. To find a point halfway in between, we note that the line through these two points is parametrized by $(5,1,0)t+(1,-1,2)(1-t) = (4,2,-2)t+(1,-1,2)$; so that putting $t = 0$ gives $(1,-1,2)$ and putting $t = 1$ gives $(5,1,0)$; so our desired point is when $t = 1/2$; which is $(2,1,-1)+(1,-1,2) = (3,0,1)$.

Problem 4. (10 points). The first step is the multiply the top equation by 2 and subtract that from the second equation; resulting in $(d-10)y-z = 0$. Thus $d = 10$ results in the equation $-z = 0$, forcing row exchange with the third row. The resulting upper triangular matrix is $\begin{pmatrix} 2 & 5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, which is indeed nonsingular. For the second part, if we put $d = 11$, then we obtain the equation $y-z = 0$, which is also the third equation, making the system singular.

Problem 5. (10 points). We have the three equations $ax+2y+3z = 0$, $ax+ay+4z = 0$, and $ax+ay+az = 0$. We see right off the bat that putting $a = 0$ removes x from every equation, so the resulting system is automatically singular. Now, applying the first step of elimination, we subtract equation 1 from equation 2 to get $(a-2)y+z = 0$, and subtract equation 1 from equation 3 to get $(a-2)y+(a-3)z = 0$. So we see that $a = 4$ makes both of these equations equal to $2y+z = 0$, and that $a = 2$ makes these equations $z = 0$ and $-z = 0$, respectively. So 0, 2, and 4 are the three values.

Problem 6. (10 points). One easily checks that $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ is a solution. To arrive at this one considers the matrix equation $\begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} =$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$. So, one first sees that $a = 1$, and then that $b + c = 0$ and $c = 1$; this gives the first two rows, the other two are obtained similarly.

Problem 7. (10 points). The obvious multiplication gives $-CA^{-1}B + D$.

Problem 8. (3+3+4+2 points).

1) We wish to find the set of vectors $\mathbf{v}$ such that $\mathbf{v} \cdot (1,1,1) = 0$. Writing $\mathbf{v}$ in components (v_1, v_2, v_3), we see that we are looking at the set of (v_1, v_2, v_3) such that $(v_1, v_2, v_3) \cdot (1,1,1) = v_1 + v_2 + v_3 = 0$; the equation of a plane.

2) Midpoints are vectors of the form $(0, \pm 1, \pm 1)$, $(\pm 1, 0, \pm 1)$, and $(\pm 1, \pm 1, 0)$, they dot with $(1,1,1)$to be zero when you have one positive and one negative component, e.g. $(0, -1, 1)$. Obviously there are six of these.

3) The "consectutive" midpoints are those which we found in part 2) (draw a picture to see why this should be so). There are six of them, and the angles between them must be sixty degrees, so that they form a planar hexagon.