18.06 FINAL SOLUTIONS

Problem 1. (10 points) $B = \begin{pmatrix} a & b & a+b \\ b & c & b+c \\ x & y & z \end{pmatrix}$. We know that symmetric matrices have real eigenvalues and orthogonal eigenvectors. So we set $x = a+b$ and $y = b+c$. This leaves only the singularity of B. For this, we note that setting $z = x+y = a+2b+c$ makes the third column a sum of the first two, thus ensuring singularity.

Problem 2. (4 points each). $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p \end{pmatrix}$.

a) To find the eigenvalues of A, we compute $det(A-\lambda I) = det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & p-\lambda \end{pmatrix} =$ $(\lambda^2)(p-\lambda)$ (because A is upper triangular). So the eigenvalues are 0 and p.

b) If $p \neq 0$, the we wish to find $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so that $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} pa \\ pb \\ pc \end{pmatrix}$. But $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ pc \end{pmatrix}$, so we need $b = pa$ and $c = pb$; and so $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ p^2 \end{pmatrix}$ works.

c) The singular vaues of A are found by first computing the eigenvalues of $A^TA = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & p \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1+p^2 \end{pmatrix}$. As this is upper triangular, the eigenvalues are 0, 1 and $1+p^2$. So the singular values are 0, 1 and $\sqrt{1+p^2}$.

d) In general, $d\mathbf{u}/dt = B\mathbf{u}$ is solved by $e^{Bt}\mathbf{u}(0)$. For us, $B = \begin{pmatrix} 2009 & 1 & 0 \\ 0 & 2009 & 1 \\ 0 & 0 & 2009+p \end{pmatrix} =$ $A + 2009I$. To compute e^{Bt}, we note that $e^{Bt} = exp(t \begin{pmatrix} 2009 & 0 & 0 \\ 0 & 2009 & 0 \\ 0 & 0 & 2009+p \end{pmatrix} + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}) =$ $\begin{pmatrix} e^{2009} & 0 & 0 \\ 0 & e^{2009} & 0 \\ 0 & 0 & e^pe^{2009} \end{pmatrix} exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$. To compute $exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$, we note that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = 0$. So $exp(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}) = I +$ $t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + (t^2/2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, and finally $e^{Bt} = \begin{pmatrix} e^{2009} & te^{2009} & e^{2009}t^2/2 \\ 0 & e^{2009} & te^{2009} \\ 0 & 0 & e^pe^{2009} \end{pmatrix}$.

So our answer is $\begin{pmatrix} e^{2009} & te^{2009} & e^{2009}t^2/2 \\ 0 & e^{2009} & te^{2009} \\ 0 & 0 & e^pe^{2009} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2009} \\ 0 \\ 0 \end{pmatrix}$.

Problem 3. (8 points) A is 4×4 and has singular values $\{3,2,1,0\}$. As the product of the singular values is (up to sign) the determinant, we get that $det(A) = 0$, so A has nontrivial nullspace, and so 0 is an eigenvalue.

Problem 4. (3 points each). $A = QR$ where Q is orthogonal and R is upper triangular with 1's on the diagonal.

a) $det(A^T A) = det(R^T Q^T QR) = det(R^T R) = det(R)^2 = 1$ since $det(R) = 1$ by the assumption on R.

b) The equation $A^T A = R^T R$ tells us that $(R^{-1})^T (A^T A) = R$; and since $(R^{-1})^T$ is lower triangular, this is exactly the elimination of $A^T A$. So the pivots are all equal to 1.

c) Yes, since $Q^{-1}(QR)Q = RQ$.

Problem 5. (10 points) $C = A^{-1}BX$. We know that similar matrices have the same eigenvalues, so putting $X = A$ forces C and B to have the same eigenvalues.

Problem 6. (4 points each) A is 3×3 and has four 0's and five 1's.

a) A has rank 0 is impossible- it isn't the zero matrix.

b) A has rank 2: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

c) A has rank 3: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Problem 7. (10 points each) A is 100×100.

a) A has all even integers as entries. Therefore each column of A has the form $2\mathbf{c}$ where $\mathbf{c}$ is a vector of integers. So we set $C = (1/2)A$. Then $det(A) = det(2C) = 2^{100} det(C)$; so $det(A)$ is an even integer (note that $det(C)$ really is an integer because all the entries of C are integers and the det can be computed by the big formula).

b) This time, we use the big formula to compute $det(C) = \sum sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$. This is a sum containing 100! terms. Now, 100! is an even number, and each term in the sum is odd (as a product of odd integers). Since the sum of two odd numbers is even, the sum of an even number of odd numbers is even; so this sum is an even integer.

Problem 8. (5 points each) We consider the vector space V of functions of the form $c_1 + c_2 e^x + c_3 e^{2x}$, with basis $\{1, e^x, e^{2x}\}$.

a) d/dx takes V to the space W spanned by $\{e^x, e^{2x}\}$. We have that $\frac{d}{dx}(1) = 0$, $\frac{d}{dx}e^x = e^x$, and $\frac{d}{dx}e^{2x} = 2e^{2x}$. So the linear transformation of d/dx in the given bases is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) We conisder the transformation ϕ from V to $\mathbb{R}$ defined by $f \to f(7)$. This is linear: $\phi(f+g) = (f+g)(7) = f(7) + g(7) = \phi(f) + \phi(g)$, and $\phi(cf) \to cf(7) = c\phi(f)$.

c) No. $\int_0^x 1 = x$ is a function not in V.