18.06 Professor Johnson Quiz 3 December 3, 2007

SOLUTIONS

1 (20 pts.) True or false. Explain why if *false*, or give an example if *true*.

(a) There exist matrices $A \neq 0$ that are simultaneously Hermitian ($A = A^H$) and unitary ($A^H = A^{-1}$).

(b) There exist matrices $A \neq 0$ that are simultaneously anti-Hermitian ($A = -A^H$) and unitary ($A^H = A^{-1}$).

(c) There exist matrices $A \neq 0$ that are simultaneously Hermitian ($A = A^H$) and anti-Hermitian ($A = -A^H$).

(d) There exist matrices A that are simultaneously Hermitian and Markov.

Solution:

(a) True. For example, $A = I, -I$, or more generally, $A = S\Lambda S^H$, where S is any unitary matrix, and Λ is a diagonal matrix whose diagonal entries are ± 1.

(b) True. For example, the 1×1 matrix $A = i, -i$, or more generally, $A = S\Lambda S^H$, where S is any unitary matrix, and Λ is a diagonal matrix whose diagonal entries are $\pm i$.

(c) False. If A is Hermitian then all the eigenvalues are real, and if it is anti-Hermitian then the eigenvalues are imaginary, and the eigenvalues cannot be at the same time real and imaginary unless they are zero. The only Hermitian matrix whose eigenvalues are all 0 is the zero matrix, but $A \neq 0$.

(d) True, e.g. $A = I$. All 2×2 examples are of the form $\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ with $0 \leq a \leq 1$.

2 (30 pts.) Suppose we form a sequence of real numbers f_k defined by the recurrence $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, starting with the initial conditions $f_0 = 2$, $f_1 = 1$ and $f_2 = 0$.

(a) Define a 3-component vector $\vec{g}_k = (f_k, f_{k-1}, f_{k-2})^T$ and a 3×3 matrix A so that the recurrence is $\vec{g}_{k+1} = A\vec{g}_k$.

(b) If you constructed A correctly, the three eigenvalues should be 1 and $\pm i$ [I'm giving you these so you *don't* have to solve a cubic equation], and the latter two eigenvectors should be $(-1, \pm i, 1)^T$. Check that you have these $\pm i$ eigenvalues and eigenvectors, and find the $\lambda = 1$ eigenvector.

(c) Give an explicit formula for f_k for any k. (By "explicit," I mean involving elementary arithmetic and powers of complex numbers only. Formulas involving A^k are not acceptable.)

(d) Is there any choice of initial conditions that will make $|f_k|$ diverge as $k \to \infty$? Explain.

Solution

(a) This recurrence gives $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. That is, the first row of A gives $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, while the second and third rows of A just give $f_k = f_k$ and $f_{k-1} = f_{k-1}$ (copying the first and second rows of $\vec{g}_k$ to the second and third rows of $\vec{g}_{k+1}$.

(b) We need to find the nullspace of $A - \lambda I$, via elimination to obtain row-reduced echelon form. In each case, it will be convenient to swap the first two rows, which will make the first pivot 1 and will not change the nullspace. For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

for which the nullspace vector is $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

To check the provided $\pm i$ eigenvectors, we just multiply them by A:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \pm i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \mp i + 1 \\ -1 \\ \pm i \end{pmatrix} = \begin{pmatrix} \mp i \\ -1 \\ \pm i \end{pmatrix} = \pm i \begin{pmatrix} -1 \\ \pm i \\ 1 \end{pmatrix}.$$

For your edification, if we had to solve for the $\pm i$ eigenvectors we would do it by elimination too, of course. For $\lambda_2 = i$: $A - iI = \begin{pmatrix} 1-i & -1 & 1 \\ 1 & -i & 0 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -i & 0 \\ 1-i & -1 & 1 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -i & 0 \\ 0 & \boxed{i} & 1 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -i & 0 \\ 0 & \boxed{i} & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -i \\ 0 & 0 & 0 \end{pmatrix}$, for which the nullspace vector is $\vec{x}_2 = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$. For $\lambda_3 = -i$, the eigenvector is just the complex conjugate $\vec{x}_3 = \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}$.

(c) We have to expand the initial vector in the eigenvectors (note that the initial vector is $\vec{g}_2$, not $\vec{g}_0$, here). There are several ways to do this. First, we can do this by inspection: you might guess that you have to add $\vec{x}_2$ and $\vec{x}_3$ to cancel the i factors, and once you guess this the other coefficients are easy:

$$\vec{g}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \left[\begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \right].$$

More explicitly, we can solve the linear system $S\vec{c} = \vec{g}_2$ for the coefficients $\vec{c}$, where S is the matrix of eigenvectors. Via elimination on the augmented matrix, we obtain $\begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 1 & i & -i & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \to$

$$\begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1+i} & 1-i & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 1+i & 1-i & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & \boxed{-2i} & -i \end{pmatrix},$$ where we have swapped rows to keep the pivots real (which simplifies the algebra somewhat). The resulting trangular system is easily solved for $\vec{c} = (1, 1/2, 1/2)^T$.

Common mis-step: Many students correctly wrote out the solution as $A^k\vec{g}_2 = S\Lambda^k S^{-1}\vec{g}_2$, but then got stuck because they tried to directly compute S^{-1}, which is painful. In linear algebra, explicitly inverting a matrix is usually a mistake, if what we want at the end is a vector! We have emphasized that you instead should solve the linear system (i.e. expand the initial vector in the eigenvectors). (On the other hand, if you just stopped at $S\Lambda^k S^{-1}$, you only lost a few points.)

Anothe common mistake: Many studends wrote $A^k = S\Lambda^k S^{-1}$, but then wrote $S^{-1} = S^H$. This is not true unless S is unitary, i.e. it has orthonormal rows. This is not true here, and there is no reason for it to be true since A is not Hermitian or unitary, etc.

To get $\vec{g}_{k+2} = A^k\vec{g}_2$, we just multiply each eigenvector by λ^k, and take the third row to get f_k:

$$f_k = 1 + \frac{1}{2}\left[i^k + (-i)^k\right] = 1 + \cos(k\pi/2).$$

(This is just the sequence $2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots$ repeated over and over.)

(d) No, because all of the eigenvalues have $|\lambda| = 1$, hence their powers don't blow up. (However, as one may check, the matrix is not unitary.)

3 (30 pts.) (a) Suppose $A = e^{iB}$ where B is Hermitian; what is $A^H A$? Hence A is a ____________ matrix.

(b) For the recurrence relation $\vec{f}_{k+1} = e^{iB}\vec{f}_k$, what is $\|\vec{f}_k\|^2/\|\vec{f}_0\|^2$? [Hint: part (a) is useful.]

(c) Compute $\vec{f}_k$ explicitly [i.e. no matrix exponentials or powers of matrices] for $B = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$ and $\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The eigenvectors of this B are $\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$, respectively.

(d) Check that your answer from (b) is true for your answer from (c).

Solution:

(a) $A^H = e^{(iB)^H} = e^{-iB^H} = e^{-iB}$. Hence $A^H A = e^{-iB}e^{iB} = e^{-iB+iB} = e^0 = I$. (Note that iB and $-iB$ obviously commute, which is why we can combine the exponentials like this.) Hence A is unitary.

Common mistake: many students forgot to take the complex conjugate, i.e. forgetting to replace i with $-i$.

(b) As in class, $\vec{f}_k = A^k\vec{f}_0$. Hence

$$\|\vec{f}_k\|^2 = \vec{f}_k^H\vec{f}_k = \vec{f}_0^H(A^k)^H A^k\vec{f}_0 = \vec{f}_0^H A^H A^H \cdots A^H A \cdots AA\vec{f}_0 = \vec{f}_0^H\vec{f}_0 = \|\vec{f}_0\|^2$$

[using the result from part (a) to cancel the $A^H A$ factors in the middle], and hence $\|\vec{f}_k\|^2/\|\vec{f}_0\|^2 = 1$. Equivalently, the product of unitary matrices is unitary, so A^k is unitary, so it preserves lengths.

(c) We first have to expand the initial condition in terms of the eigenvectors. This is easy enough to do by inspection here:

$$\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -1 \end{pmatrix}}{5}.$$

Alternatively, we could solve the 2×2 system $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the coefficients $c_1 = 1/5$ and $c_2 = 2/5$. Or, we could use the orthogonality to get $c_j = \vec{f}_0 \cdot \vec{x}_j / \|\vec{x}_j\|^2$. Once this is done, we use the fact that $\vec{f}_k = A^k \vec{f}_0 = e^{iBk}\vec{f}_0$ to multiply each eigenvector by $e^{i\lambda k}$:

$$\vec{f}_k = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{i5k} + 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-i5k}}{5} = \frac{\begin{pmatrix} e^{i5k} + 4e^{-i5k} \\ 2e^{i5k} - 2e^{-i5k} \end{pmatrix}}{5}.$$

(d) This is simplest if we don't combine the terms above and instead use the orthogonality to eliminate the $\vec{x}_1 \cdot \vec{x}_2$ and $\vec{x}_2 \cdot \vec{x}_1$ cross terms:

$$\|\vec{f}_k\|^2 = \vec{f}_k^H \vec{f}_k = \frac{\left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\|^2 \left|e^{i5k}\right|^2 + 2^2 \left\|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\|^2 \left|e^{-i5k}\right|^2}{5^2} = \frac{5 + 4 \cdot 5}{25} = 1 = \|\vec{f}_0\|^2.$$

Alternatively, we can explicitly write out

$$\begin{aligned} |e^{i5k} + 4e^{-i5k}|^2 + |2e^{i5k} - 2e^{-i5k}|^2 &= (e^{i5k} + 4e^{-i5k})(e^{-i5k} + 4e^{i5k}) + (2e^{i5k} - 2e^{-i5k})(2e^{-i5k} - 2e^{i5k}) \\ &= (1 + 4e^{-i10k} + 4e^{i10k} + 16) + (4 - 4e^{-i10k} - 4e^{i10k} + 4) \\ &= 25, \end{aligned}$$

so again $\|\vec{f}_k\|^2 = 25/25 = 1 = \|\vec{f}_0\|^2$.

4 (20 pts.) Some 3×3 real matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, with the corresponding eigenvectors $\vec{x}_1 = (1,0,0)^T$, $\vec{x}_2 = (0,1,2)^T$, and $\vec{x}_3 = (0,1,1)^T$.

(a) Give a basis for: (i) the nullspace $N(A)$, (ii) the column space $C(A)$, and (iii) the row space $C(A^H)$.

(b) Find all solutions $\vec{x}$ to $A\vec{x} = \vec{x}_2 - 3\vec{x}_3$.

(c) Is A (i) real-symmetric, (ii) orthogonal, (iii) Markov, or (iv) none of the above?

Solution:

(a) The nullspace is just the span of the $\lambda = 0$ eigenvector $\vec{x}_1$. If we act A on any vector, we only get multiples of the $\lambda \neq 0$ eigenvectors, so $C(A)$ is the span of $\vec{x}_2$ and $\vec{x}_3$. The row space is the orthogonal complement of the nullspace, and here this is spanned by (e.g.) the vectors $(0,1,0)^T$ and $(0,0,1)^T$.

(b) The right hand side is clearly in the column space. Since we have expanded the right hand side in the $\lambda \neq 0$ eigenvectors, we can get a particular solution just by dividing them by the corresponding eigenvalues: remember, A acts just like a number on these vectors. Hence a particular solution is $\vec{x}_p = \vec{x}_2/1 - 3\vec{x}_3/2 = (0,-1/2,1/2)^T$. To get all the solutions we must add the nullspace, obtaining $\vec{x} = (a,-1/2,1/2)^T$ for any constant a.

Equivalently, expand $\vec{x}$ in the eigenvectors, $\vec{x} = a\vec{x}_1 + b\vec{x}_2 + c\vec{x}_3$, and plug in to $A\vec{x} = b\vec{x}_2 + 2c\vec{x}_3 = \vec{x}_2 - 3\vec{x}_3$ to find $a =$ arbitrary, $b = 1$, and $c = -3/2$.

(c) (iv) None of the above. It's clearly not Markov or orthogonal since there is a $\lambda = 2$ eigenvalue. Although the eigenvalues are real, it's not real-symmetric since the eigenvectors are not orthogonal.