SOLUTIONS

1 (20 pts.)   Find all solutions to the linear system

$$x + 2y + z - 2w = 5$$

$$2x + 4y + z + w = 9$$

$$3x + 6y + 2z - w = 14$$

Solution:

We perform elimination on the augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & -2 & 5 \\ 2 & 4 & 1 & 1 & 9 \\ 3 & 6 & 2 & -1 & 14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -2 & 5 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 5 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -2 & 5 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

So y and w are free variables. Thus special solutions to $A\mathbf{x} = 0$ are given by setting $y = 1, w = 0$ and $y = 0, w = 1$ respectively, i.e.

$$\mathbf{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{s}_2 = \begin{pmatrix} -3 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

Moreover, a particular solution to the system is given by setting $y = w = 0$, i.e.

$$\mathbf{x}_p = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We could have read these special and particular solutions off even more easily by performing one more elimination step to get the row-reduced echelon matrix:

$$\rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

Notice that the last column gives the values of the pivot variables for the particular solution, and the free columns give the values of the pivot variables in the special solutions (multiplied by -1), as was shown in class.

We conclude that the general solutions to this system are given by

$$\mathbf{x} = \mathbf{x}_p + c_1 \mathbf{s}_1 + c_2 \mathbf{s}_2 = \begin{pmatrix} -2c_1 - 3c_2 + 4 \\ c_1 \\ 5c_2 + 1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

2 (30 pts.) In class, we learned how to do "downwards" elimination to put a matrix A in upper-triangular (or echelon) form U: not counting row swaps, we subtracted multiples of pivot rows from subsequent rows to put zeros below the pivots, corresponding to multiplying A by elimination matrices.

Instead, we could do elimination "leftwards" by subtracting multiples of pivot columns from leftwards columns, again to get an upper-triangular matrix U. For example, let:

$$A = \begin{pmatrix} 7 & 6 & 4 \\ 6 & 3 & 12 \\ 2 & 0 & 1 \end{pmatrix}$$

We could subtract twice the third column from the first column to eliminate the 2, so that we get zeros to the left of the "pivot" 1 at the lower right.

(i) Continue this "leftwards" elimination to obtain an upper-triangular matrix U from the A above, and write U in terms of A multiplied by a sequence of matrices corresponding to each leftwards-elimination step.

(ii) Suppose we followed this process for an arbitrary A (not necessarily square or invertible) to get an echelon matrix U. Which of the column space and null space, if any, are the same between A and U, and why?

(iii) Is the U that we get by leftwards elimination always the same as the U we get from ordinary downwards elimination? Why or why not?

Solution:

(i) The "leftwards" elimination procedure is

$$A = \begin{pmatrix} 7 & 6 & 4 \\ 6 & 3 & 12 \\ 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 6 & 4 \\ -18 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 35 & 6 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} = U,$$

where the first step sent (col1) → (col1) − 2(col3) and the second step sent (col1) → (col1) + 6(col2). Since these operations are linear combinations of the columns, they correspond to multiplying on the *right* by elimination matrices:

$$U = \begin{pmatrix} 35 & 6 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 1 \end{pmatrix} = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(One way to get these elimination matrices, as usual, is to do the corresponding operation on the 3×3 identity matrix.)

(ii) As above, $U = AE$ (the elimination matrices multiply on the *right*). It follows that the column space of A is the same as the column space of U, but the null spaces are different. Informally, by multiplying E on the *right*, we modify the *input* vectors of A (changing the null space), but the output vectors are still made of columns of A (preserving the column space). To be more careful, we need the fact that E is invertible (as elimination always is); otherwise, $C(AE)$ could be a smaller subspace of $C(A)$.

More precisely, since $U = AE$ above, where E are the elimination matrices, any $\mathbf{x} = U\mathbf{y} = A(E\mathbf{y})$, so any $\mathbf{x}$ in $C(U)$ is in $C(A)$. Also vice-versa, since $A = UE^{-1}$. So $C(U) = C(A)$. However, if $\mathbf{x}$ is in the null space $N(U)$ (i.e. $U\mathbf{x} = \mathbf{0} = AE\mathbf{x}$), this only means $E\mathbf{x}$ is in $N(A)$, not $\mathbf{x}$. So the null spaces are different in general (but have the same dimension). [Compare to the case of ordinary elimination, which preserves $N(A)$ but changes $C(A)$. Left elimination is equivalent to "upwards" elimination on A^T—this preserves the row space of A^T, meaning that the column space of A is preserved etc.]

(iii) The are *not* the same U in general (although of course there are special cases where they are the same, such as when A is upper-triangular to start with). There are several ways to see this.

The simplest way is to give any counterexample: e.g., apply downwards elimination to A above and you will get a different result. For example, downward elimination never

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changes the upper-left corner (7), but the upper-left corner was changed (to 35) by leftwards elimination above.

Abstractly, we know from class that downwards elimination always preserves the null space, whereas we just saw that upwards elimination does not. So, they cannot be the same.

It is *not* sufficient to simply say that left-elimination does different sorts of operations than down-elimination—there are lots of problems where you can do a different sequence of operations and still get the same result. (For example, we could use left elimination to find A^{-1}, and of course there is only one possible A^{-1} if it exists at all.)

3 (20 pts.) Determine whether the following statements are true or false, and explain your reasoning.

(♣) If $A^2 = A$, then $A = 0$ or $A = I$.

(♢) Ignoring row swaps, any invertible matrix A has a "UL" factorization (as an alternative to LU factorization): A can be written as $A = UL$ where U and L are some upper and lower triangular matrices, respectively.

(♠) All the 2×2 matrices that commute with $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ (i.e. all 2×2 matrices B with $AB = BA$) form a vector space (with the usual formulas for addition of matrices and multiplication of matrices by numbers).

(♡) There is no 3×3 matrix whose column space equals its nullspace.

Solution

(♣) False.

Counterexample: if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $A^2 = A$ but $A \neq I$ and $A \neq 0$.

Another counterexample was given in Pset 2 Problem 8 (a). Note that if we assume A is invertible, then the only solution is $A = I$ (multiply both sides of $A^2 = A$ by A^{-1}), but this assumption is not warranted here.

(♢) True.

Instead of "downwards" elimination we can also do "upwards" elimination to put A into lower-triangular form L (possibly with row swaps). In this procedure the corresponding elimination matrices are upper-triangular, but still multiply on the left (since they are still row operations), so we get a UL factorization.

Alternatively, the "leftwards" elimination of problem 2 also leads to a UL factorization, because the (lower-triangular!) elimination matrices multiply on the *right* to give $U = AL^{-1}$.

Technically, however, this process may require *column* swaps if zeros are encountered in pivot positions.

(♠) True.

We need to know that linear combinations of vectors stay in the vectors space. If B is a matrix where $AB = BA$, then clearly $A(cB) = c(AB) = c(BA) = (cB)A$ for any c. If B' is another matrix where $AB' = B'A$, then $A(B + B') = AB + AB' = BA + B'A = (B + B')A$. (The other properties of a vector space, associativity etcetera, need not be shown since they are automatic for the usual addition and multiplication operations.)

(♡) True.

Suppose the rank of A is r, then the dimension of column space is r, and the dimension of null space is $3-r$. Obviously no matter $r = 0, 1, 2, 3$, we always have $r \neq 3-r$. (Equivalently, $r = 3 - r$ would imply a fractional rank $r = 3/2$!) This shows that the two spaces are not the same, since they must have different dimensions.

4 (30 pts.) The following information is known about an $m \times n$ matrix A:

$$A\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, A\begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, A\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, A\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(α) Show that the vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ form a basis of $\mathbb{R}^4$.

(β) Give a matrix C and an invertible matrix B such that $A = CB^{-1}$. (You *don't* have to evaluate B^{-1} or find A explicitly. Just say what B and C are and use them to reason about A in the subsequent parts.)

(γ) Find a basis for the null space of A^T.

(δ) What are m, n, and the rank r of A?

Solution:

(α) We are in $\mathbb{R}^4$, which is four-dimensional, so any four linearly independent vectors forms a basis as shown in class. Thus, we just need to show that these four vectors are linearly independent, which is equivalent to showing that the 4×4 matrix whose columns are these vectors has full column rank (null space = $\{\mathbf{0}\}$). Proceeding by elimination:

$$B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -7 & -15 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix} = U.$$

Thus, there are four pivots, and hence it has full column rank as desired.

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(β) The provided equations multiply A by four vectors to get four vectors, which by definition of matrix multiplication (recall the column picture) can be combined into a single equation where A is multiplied by a matrix with four columns to yield a matrix with four columns:

$$A \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{pmatrix}.$$

Thus if we take

$$C = \begin{pmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

we have $A = CB^{-1}$. Since B is precisely the matrix of the basis vectors from part (α), its invertibility follows from above (it is 4×4 and has 4 pivots).

(γ) Since $A = CB^{-1}$, we have

$$A^T = (B^{-1})^T C^T = (B^T)^{-1} C^T.$$

(As in class, because B is invertible, B^T is too.) Just as for elimination (multiplying on the left by an invertible elimination matrix), the null space is preserved when $C^T \to (B^T)^{-1}C^T$. [You need not prove this, because the proof is the same as in class. Recall that if $C^T \mathbf{x} = \mathbf{0}$ then $A^T \mathbf{x} = \mathbf{0}$ from above, and vice versa if we multiply both sides by B^T.] That means we just need to find the null space of C^T by elimination:

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\begin{pmatrix} 2 & 4 \\ 0 & 0 \\ 5 & 10 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

in which there is only one free variable, so there is one special solution (the basis of the null space)

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\mathbf{s}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},
$$

or any multiple thereof. (You can also find this special solution by inspection, without elimination.)

(δ) Since A times a 4-vector is a 2-vector, we must have $m = 2$ and $n = 4$. Equivalently, from part (β) we saw that A was a 2×4 matrix multiplied by a 4×4 matrix, giving a 2×4 matrix. Moreover, from above the dimension of $N(A^T)$ is 1, but this must equal $m - r$, so we obtain $r = 1$.