18.06 Problem Set 7

Due Wednesday, 07 November 2007 at 4 pm in 2-106.

Problem 1: Consider the matrix $A = \begin{pmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{pmatrix}$.

(a) If one eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T$, find its eigenvalue λ_1.
(b) Show that $\det(A) = 0$. Give another eigenvalue λ_2, and find the corresponding eigenvector $\mathbf{v}_2$?
(c) Given the eigenvalue $\lambda_3 = 4$, write down a linear system which can be soloved to find the eigenvector $\mathbf{v}_3$.
(d) What is the trace of A? Use this to find λ_4.

Problem 2: (a) Suppose $n \times n$ matrices A, B have the same eigenvalues $\lambda_1, \cdots, \lambda_n$, with the same independent eigenvectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$. Show that $A = B$.
(b) Find the 2×2 matrix A having eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
(c) Find two different 2×2 matrices A, B, both have the same eigenvalues $\lambda_1 = \lambda_2 = 2$, and both have the same eigenvector (only one) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
(d) Find a matrix which has two different sets of independent eigenvectors.

Problem 3: Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors of A.
(b) Calculate A^{100} (not by multiplying A 100 times!).
(c) Find all eigenvalues and corresponding eigenvectors of $A^3 - A + I$.
(d) For any polynomial function f, what are the eigenvalues and eigenvectors of $f(A)$? Prove your statement.

Problem 4: For simplicity, assume that A has n independent eigenvectors, thus diagonalizable. (However, the statement below also holds for general matrices.)
(a) Since A is diagonalizable, we can write $A = S\Lambda S^{-1}$ as in class. Substitute this into the characteristic polynomial

$$p(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I)$$

to show that $p(A) = 0$. (This is called the **Cayley-Hamilton theorem**.)

(b) Test the Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then write A^{-1} as a polynomial function of A. [Hint: move the I term to one side of the equation $p(A) = 0$ to write $A \cdot (\text{something}) = I$.]

(c) Use the Cayley-Hamilton theorem above to show that, for any invertible matrix A, A^{-1} can always be written as a polynomial of A. (Inverting using elimination is usually much more practical, however!)

Problem 5: (a) If A (an $n \times n$ matrix) has n nonnegative eigenvalues λ_k and independent eigenvectors $\mathbf{x}_k$, and if we define "$\sqrt{A}$" as the matrix with eigenvalues $\sqrt{\lambda_k}$ and the same eigenvectors, show that $(\sqrt{A})^2 = A$.

(b) Given $\sqrt{A}$ from part (a), is the only other matrix whose square is A given by $-\sqrt{A}$? Why or why not?

Problem 6: If λ is an eigenvalue of A, is it also an eigenvalue of A^T? What about the eigenvectors? Justify your answers.

Problem 7: This problem refers to *similar matrices* in the sense defined by section 6.6 (page 343) of the text.

(a) A, B, and C are square matrices where A is similar to B and B is similar to C. Is A similar to C? Why or why not?

(b) If A is similar to Q, where Q is an orthogonal matrix ($Q^T Q = I$), is A orthogonal too? Why or why not?

(c) If A is similar to U where U is triangular, is A triangular too? Why or why not?

(d) If A is similar to B where B is symmetric, is A symmetric too? Why or why not?

(e) For a given A, does the set of all matrices similar to A form a subspace of the set of all matrices (under ordinary matrix addition)? Why or why not?

Problem 8: The *Pell numbers* are the sequence $p_0 = 0$, $p_1 = 1$, $p_n = 2p_{n-1} + p_{n-2}$ $(n > 1)$.

(a) Analyzing the Pell numbers in the same way that we analyzed the Fibonacci sequence, via eigenvectors and eigenvalues of a 2×2 matrix, find a closed-form expression for p_n.

(b) Prove that the ratio $(p_{n-1} + p_n)/p_n$ tends to $\sqrt{2}$ as n grows.

(c) Using Matlab or a calculator, evaluate the ratio from (b) up to $n = 10$ or so, and subtract $\sqrt{2}$ to find the difference for each n. Compare this method of computing $\sqrt{2}$ to Newton's method from 18.01:[1] starting with $x = 1$, repeatedly replace x by $(x + 2/x)/2$. Which technique goes faster to $\sqrt{2}$?

Problem 9: This is a Matlab problem on symmetric matrices.
(a) Use Matlab to construct a random 4×4 symmetric matrix (`A = rand(4,4); A = A' * A`), and find its eigenvalues via the command `eig(A)`.
(b) Construct 1000 random vectors as the columns of X via `X = rand(4,1000) - 0.5;` and for each vector $\mathbf{x}_k$ compute $d_k = \mathbf{x}_k^T A \mathbf{x}_k / \mathbf{x}_k^T \mathbf{x}_k$ via the command `d = diag(X'*A*X) ./ diag(X'*X);` (which computes a vector $\mathbf{d}$ of these 1000 ratios).
(c) Find the minimum ratio d_k via `min(d)` and the maximum via `max(d)`. How do these compare to the eigenvalues? Try increasing from 1000 to 10000 above to see if your hypothesis holds up.
(d) Find the eigenvectors of A via `[S,L] = eig(A)`: L is the diagonal matrix Λ of eigenvalues, and S is the matrix whose columns are the eigenvectors, so that $AS = SL$. Does S have any special properties? (Hint: try looking at `det(S)` and `inv(S)` compared to S.) (We will see in class, eventually, that these and other nice properties come from A being symmetric.)
(e) If you wanted to pick a vector $\mathbf{x}$ to get the maximum possible d, what would $\mathbf{x}$ and the resulting d be? What about the minimum?
(f) Repeat the process for a 2×2 complex *asymmetric* A and a *complex* X. This time, plot the resulting (complex) d_k values in the complex plane as black dots, and the (complex) eigenvalues as red circles:

```
A = rand(2,2) + i*rand(2,2);
X = rand(2,1000)-0.5+i*(rand(2,1000)-0.5);
d = diag(X'*A*X) ./ diag(X'*X);
v = eig(A)
plot(real(d), imag(d), 'k.', real(v), imag(v), 'ro')
```

You should get an elliptical region with the eigenvalues as foci.
(g) It is hard to see in part (f) that the eigenvalues are foci unless they are close together, which is unlikely. Construct a random non-symmetric A with eigenvalues $1 + 1i$ and $1.05 + 0.9i$ by performing a random 2×2 similarity transformation on `D=diag([1+1i,1.05+0.9i])` (i.e. construct a random S and multiply `A = inv(S) * D * S`), and then repeat (f) with this A.

[1] The square root $\sqrt{y}$ is the root of $f(x) = x^2 - y$, and Newton's method replaces x by $x - f(x)/f'(x) = (x + y/x)/2$. Actually, this technique for square roots was known to the Babylonians 3000 years ago.