18.06 Problem Set 6

Due Wednesday, 24 October 2007 at 4 pm in 2-106.

Problem 1: Do problem 4 from section 4.4 (P 228) in your book.

Problem 2: Apply the Gram-Schmidt algorithm to find an orthonomal basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors:

$$\mathbf{v}_1 = (1,1,1,1), \mathbf{v}_2 = (1,1,2,4), \mathbf{v}_3 = (1,2,-4,-3).$$

Write down the QR decomposition of the matrix A whose columns are these three vectors.

Problem 3: In the Gram-Schmidt algorithm, at each step we subtract the projection of one vector onto the previous vectors, in order to make them orthogonal. The key operation is the inner product $\mathbf{x}^T\mathbf{y}$, sometimes denoted $\mathbf{x}\cdot\mathbf{y}$ or $\langle\mathbf{x},\mathbf{y}\rangle$. We can apply the same process to *any* vector space as long as we define a suitable "inner product" that obeys the same algebraic rules. The key rules that a inner product must obey (for real vector spaces) are:

$$\begin{array}{ll} (a) & \langle \mathbf{x}_1+\mathbf{x}_2,\mathbf{y}\rangle = \langle \mathbf{x}_1,\mathbf{y}\rangle + \langle \mathbf{x}_2,\mathbf{y}\rangle; \\ (b) & \langle c\mathbf{x},\mathbf{y}\rangle = c\langle \mathbf{x},\mathbf{y}\rangle; \\ (c) & \langle \mathbf{x},\mathbf{y}\rangle = \langle \mathbf{y},\mathbf{x}\rangle; \\ (d) & \langle \mathbf{x},\mathbf{x}\rangle \geq 0 \text{ for all } \mathbf{x}, \text{ and } \langle \mathbf{x},\mathbf{x}\rangle = 0 \text{ if and only if } \mathbf{x}=0. \end{array}$$

In particular, instead of the vector space $\mathbf{R}^m$ of column vectors, consider instead the vector space V of real-coefficient polynomial functions $f(x)$, $g(x)$, etc.

(1) Show that $\langle f,g\rangle := \int_0^1 f(x)g(x)\,dx$ defines an inner product on V, i.e., check that it satisfies the above four properties.

(2) Show that the polynomials $f = 1+x$ and $g = 5-9x$ are orthogonal.

(3) Apply the Gram-Schmidt algorithm to the set $\{1, x, x^2\}$ to obtain an orthonormal basis $\{f_0, f_1, f_2\}$ of all degree-2 polynomials.

(4) Do the same thing *approximately* in Matlab, approximating a polynomial by its values over a set of 1000 discrete points, and the integral by a summation:

```
x = linspace(0,1,1000)';
A = [x.^0, x.^1, x.^2, x.^3, x.^4, x.^5];
```

That is, the columns of A are x^0, x^1, ..., x^5 (discretized). Matlab will do Gram-Schmidt for us via the function `qr` (passing zero as the second argument to `qr` will just do Gram-Schmidt of a non-square matrix rather than trying to construct a square orthogonal Q):

```
[Q,R] = qr(A, 0); Q = Q * sqrt(1000);
```

The $\sqrt{1000}$ factor is to change the normalization to match the approximate "integral" inner product rather than the ordinary dot product. (Why? Think about how the dot product compares to the approximate discretized integral for x^0.)[1] Now plot the columns of Q versus x in order to see the orthogonal "polynomials" up to degree 5.

```
plot(x, Q)
```

Verify that they match your answers from part (3), up to degree 2 of course, within the numerical error, by plotting the curves on top of one another. (You can superimpose plots in Matlab by typing `hold on` at the Matlab prompt, which makes subsequent plot commands plot on top of one another; to stop superimposing, type `hold off`.)

Problem 4: True or False: (Give reasons)
Suppose all matrices below are square matrices.
(1) $\det(-A) = -\det(A)$.
(2) $\det(A + B) = \det(A) + \det(B)$.
(3) $\det(A - B) = \det(A) - \det(B)$.
(4) $\det(AB) = \det(A)\det(B)$.
(5) $\det(A^{-1}) = \det(A)^{-1}$.

Problem 5: Calculate the determinant of the following 6×6 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

[1] Strictly speaking, the endpoints aren't really handled correctly from the perspective of numerical integration; e.g. in a trapezoidal rule the endpoints would have half weight. But the error introduced that way is small enough to ignore for our purposes.

Problem 6: Construct a random symmetric 5×5 matrix A in Matlab:

```
A = rand(5,5); A = A' * A
```

Compute the QR decomposition of A, via:

```
[Q,R] = qr(A);
```

(a) Verify that $Q^T Q = I$ (that Q is orthogonal) and that $QR = A$, up to roundoff error (about 10^{-16}).
(b) Compute

```
B = R * Q
```

Notice that B is symmetric. Why? (Hint: write R in terms of A.)
(c) Repeat this process. Compute:

```
[Q,R] = qr(B); B = R * Q
```

over and over again. You should find that, after repeating this a number of times, the result stops changing (if you ignore numbers smaller than 10^{-15}. Compute

```
eig(A)
```

which prints the eigenvalues of A (something you learned in 18.03, and we'll study in more detail soon). Compare this to the B resulting from the above process. How are they related?
(d) Explain why, if B ever becomes a completely diagonal matrix, it should not change at all when we do the step in part (c).