# 18.06 Problem Set 4 - Solutions

Due Wednesday, 10 October 2007 at 4 pm in 2-106.

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**Problem 1:** (10=2+2+2+2+2) Decide whether the following set of vectors are linearly dependent or independent. (Give reasons)

(a) $(1, 2, 3), (2, 3, 1), (3, 1, 2)$.

$\boxed{\text{Solution}}$ Linearly independent.

Reason: The matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{pmatrix}$$

is of full rank.

(b) $(1, 1, 0, 0), (1, -1, 0, 0), (1, 0, 0, 0)$.

$\boxed{\text{Solution}}$ Linearly dependent.

Reason: by observation $\frac{1}{2}(1, 1, 0, 0) + \frac{1}{2}(1, -1, 0, 0) = (1, 0, 0, 0)$.

(c) $(0, 0, 0), (1, 4, 5), (1, 0, 4)$.

$\boxed{\text{Solution}}$ Linearly dependent.

Reason: Any set of vectors containing the zero vector is linearly dependent.

(d) $1, 1 + x, 1 + x^2$ (in the vector space of polynomials).

$\boxed{\text{Solution}}$ Linearly independent.

Reason: suppose

$$a_1 + a_2(1 + x) + a_3(1 + x^2) = 0,$$

then

$$a_3x^2 + a_2x + (a_1 + a_2 + a_3) = 0,$$

which implies $a_3 = 0, a_2 = 0$ and $a_1 = 0$.

(e) $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_4 - \mathbf{v}_1$.

$\boxed{\text{Solution}}$ Linearly dependent.

Reason: by observation,

$$(\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) + (\mathbf{v}_3 - \mathbf{v}_4) + (\mathbf{v}_4 - \mathbf{v}_1) = 0.$$

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**Problem 2:** (10=2+2+2+2+2) Find a basis of the following vector spaces.

(a) All vectors in $\mathbb{R}^3$ whose components are equal.

$\boxed{\text{Solution}}$ Such vectors are of the form $(x, x, x)$. They form a one dimensional subspace of $\mathbb{R}^3$. A basis is given by $(1, 1, 1)$.

(Any nonzero vector $(a, a, a)$ will give a basis.)

(b) All vectors in $\mathbb{R}^4$ whose components add to zero and whose first two components add to equal twice the fourth component.

$\boxed{\text{Solution}}$ The subspace consists of vectors

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0, x_1 + x_2 = 2x_4\}.$$

It is the nullspace of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix}$, a two dimensional subspace of $\mathbb{R}^4$, so any two independent vector gives a basis. For example, we can take as a basis $v_1 = (1, 1, -3, 1), v_2 = (1, -1, 0, 0)$.

(c) All vectors in $\mathbb{R}^4$ that are perpendicular to $(1, 0, 1, 0)$.

$\boxed{\text{Solution}}$ This subspace is just the nullspace of the $1 \times 4$ matrix $(1, 0, 1, 0)$. This is a three dimensional hyperplane in $\mathbb{R}^4$. A basis can be read from the matrix (it is of row reduced echelon form already!):

$$v_1 = (0, 1, 0, 0), v_2 = (-1, 0, 1, 0), v_3 = (0, 0, 0, 1).$$

(d) All anti-symmetric $3 \times 3$ matrices.

$\boxed{\text{Solution}}$ Any anti-symmetric $3 \times 3$ matrix is of the form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Thus it is a 3 dimensional subspace, and a basis is given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

(e) All polynomials $p(x)$ whose degree is no more than 3 and satisfies $p(0) = 0$.

$\boxed{\text{Solution}}$ We can write the polynomial $p(x)$ as $p(x) = ax^3 + bx^2 + cx + d$. The condition $p(0) = 0$ implies $d = 0$. Thus the space is $\{ax^3 + bx^2 + cx \mid a, b, c \in \mathbb{R}\}$. It is a three dimensional subspace in the vector space of polynomials, and a basis is given by $p_1(x) = x^3, p_2(x) = x^2, p_3(x) = x$.

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**Problem 3:** (10) Do problem 13 from section 3.5 (P 169) in your book.

$\boxed{\text{Solution}}$ The row reduced echelon form $U$ has two pivots, thus $A$ has rank 2. Since $A$ is $3 \times 3$ matrix, we conclude

$$\dim C(A) = 2,\ \dim C(A^T) = 2,\ \dim N(A) = 3 - 2 = 1,\ \dim N(A^T) = 1.$$

Since $U$ is the row reduced echelon form of $A$, their row spaces are the same. (However, their column spaces are different. For example, $(1, 1, 3)$ lies in the column space of $A$, but not in the column space of $U$. )

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**Problem 4:** (10) Do problem 2 from section 3.6 (P 180) in your book.

$\boxed{\text{Solution}}$

For $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

thus the matrix is of rank 1, so

A basis of $C(A)$: $v = (1, 2)$.

A basis of $C(A^T)$: $v = (1, 2, 4)$.

A basis of $N(A)$: $v_1 = (-2, 1, 0), v_2 = (-4, 0, 1)$.

A basis of $N(A^T)$: $v = (-2, 1)$.

For $B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix},$$

thus the matrix is of rank 2, so

A basis of $C(B)$: $v_1 = (1, 2), v_2 = (2, 5)$.

A basis of $C(B^T)$: $v_1 = (1, 2, 4), v_2 = (2, 5, 8)$.

A basis of $N(B)$: $v_1 = (-4, 0, 1)$.

A basis of $N(B^T)$: None. (Since $N(B^T) = \{0\}$ is the 0 dimensional space.)

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**Problem 5:** (10) Do problem 22 from section 3.6 (P 182) in your book.

$\boxed{\text{Solution}}$ Since the row space has basis $(1, 0, 0), (0, 1, 1)$, we see that each row is of the form $(a, b, b)$, in other words, the the second column vector of $A$ must equals the third column vector. We may take $u, w$ to be any two independent vector in the subspaces spanned by $(1, 2, 4), (2, 2, 1)$, and $v, z$ to be the two given row vector, then the matrix $A$ satisfies the conditions.

For example, we may take

$$u = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

then

$$A = uv^T + wz^T = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{pmatrix},$$

which obviously satsifies the conditions.

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**Problem 6:** (10) Do problem 23 from section 3.6 (P 182) in your book.

$\boxed{\text{Solution}}$ Denote the given product by $A = LR$, then the rows of $A$ are linear combination of rows of $R$, and the columns of $A$ are linear combinations of columns of $L$. Thus:

A basis of the row space of $A$ is given by $u_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

A basis of the column space of $A$ is given by $v_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$.

$A$ is not invertible since it is a $3 \times 3$ matrix but $rank(A)$ is 2.

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**Problem 7:** (10=2+2+2+2+2) True or false: (Give reasons)

(a) If the row space of $A$ equals the column space of $A$, then $A^T = A$.

$\boxed{\text{Solution}}$ False.

Counterexample: take $A$ to be any nonsingular matrix, then both the row space of $A$ and the column space of $A$ are the full space, but $A$ don't have to be symmetric.

(Other examples: take $A$ to be any anti-symmetric matrix, maybe singular, then we also have $C(A) = C(A^T)$.)

(b) If the column vectors of a matrix are dependent, so are the row vectors.

$\boxed{\text{Solution}}$ False.

Counterexample: take $A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, then the columns are dependent 1-vectors, but it only has 1 row, so row vector is independent. More generally, we may take $A$ to be any $m \times n$ $(m < n)$ matrix with rank $m$. Then the column are dependent since $rank(A) < n$, but the rows are independent since $rank(A) = m$.

However, if $A$ is $n \times n$ square matrix, then the dependence of column vectors implies the dependence of row vectors, and vice versa.

(c) The matrices $A$ and $-A$ have the same four subspaces.

$\boxed{\text{Solution}}$ True.

Since a vector space contains a vector $v$ if and only if it contains the vector $-v$, we see $C(A) = C(-A)$.

Since $Ax = 0$ if and only if $-Ax = 0$, we see $N(A) = N(-A)$.

Apply the above arguments to $A^T$ and $-A^T$, we see $C(A^T) = C((-A)^T)$ and $N(A^T) = N((-A)^T)$.

(d) The rows of a matrix are a basis of the row space.

$\boxed{\text{Solution}}$ False.

Counterexample: the matrix $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has 2 rows, and the row space is 1-dimensional, thus a basis of the row space will only consists 1 vector.

In fact, the rows of a matrix form a basis of the row space if and only if the matrix is of full row rank.

(e) The column space of a $3 \times 4$ matrix has the same dimension as its row space.

$\boxed{\text{Solution}}$ True.

No matter what the matrix is, the column space will always has the same dimension as its row space, both equal the rank of the matrix.

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**Problem 8:** (10=4+4+2) Give the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{pmatrix}$ depending on $c$.

(a) find a basis for the column space of $A$.

$\boxed{\text{Solution}}$ Elimination gives

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{pmatrix},$$

so there are two cases:

- If $c \neq 3$, then $c-3$ is a pivot. From the position of pivots we see that a basis of $C(A)$ is given by the first three columns of $A$:

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

- If $c = 3$, then we can do further elimination

$$A \to \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

From the position of pivots we see that a basis of $C(A)$ is given by

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

(b) find a basis for the nullspace of $A$.

$\boxed{\text{Solution}}$ We have two cases as before.

- If $c \neq 3$, then there is only one free variable $x_4$, and the special solution is given by $\mathbf{s} = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

- If $c = 3$, then there are two free variables $x_2$ and $x_4$. The special solutions are given by $\mathbf{s}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

(c) find the complete solution $x$ to $A\mathbf{x} = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$.

$\boxed{\text{Solution}}$ Since the vector $\begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}$ is just the second column of the matrix, by inspection one see that $x_p = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a paticular solution.

- If $c \neq 3$, then the complete solution is $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.
- If $c = 3$, then the complete solution is $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

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**Problem 9:** (10) Do problem 8 from section 8.2 (P 421) in your book. (Note that the graph is the second one on page 420.)

$\boxed{\text{Solution}}$ From the graph, we can write down the incident matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

A maximal subtree of the graph contains three edges (e.g. edge 1, edge 2 and edge 5). So the rank of $A$ is 3.

The nullspace is $4 - 3 = 1$ dimensional, which contains the vector $(1, 1, 1, 1)$. Thus $\mathbf{u} = (1, 1, 1, 1)$ is a solution to $A\mathbf{x} = 0$.

The left nullspace is $5 - 3 = 2$ dimensional, and can be find by tracing the loops. The first loop is given by edges $1, 3, -2$, which gives a vector $\mathbf{v}_1 = (1, -1, 1, 0, 0)$. The second loop is given by the edges $3, 5, -4$, gives the vector $\mathbf{v}_2 = (0, 0, 1, -1, 1)$. They are two solutions to $A^T y = 0$.

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**Problem 10:** (10=4+2+4) (a) Use MATLAB to construct a random 10×5 matrix $A$ and a random 5×9 matrix $B$. Then use MATLAB to find bases for the four subspaces of the matrix $AB$. (Hints: The commands $A = rand(10, 5); B = rand(5, 9)$; will give you the two random matrix, and the command $[R, p] = rref(A)$ returns the row-reduced echelon form R of A and a list p of the pivot columns.)

$\boxed{\text{Solution}}$ The inputs and outputs are

```
>> A=rand(10,5)
A =
0.6476 0.4587 0.5822 0.5447 0.4046
0.6790 0.6619 0.5407 0.6473 0.4484
0.6358 0.7703 0.8699 0.5439 0.3658
0.9452 0.3502 0.2648 0.7210 0.7635
0.2089 0.6620 0.3181 0.5225 0.6279
0.7093 0.4162 0.1192 0.9937 0.7720
0.2362 0.8419 0.9398 0.2187 0.9329
0.1194 0.8329 0.6456 0.1058 0.9727
0.6073 0.2564 0.4795 0.1097 0.1920
0.4501 0.6135 0.6393 0.0636 0.1389
>> B=rand(5,9)
B =
0.6963 0.4849 0.3477 0.7549 0.7363 0.0196 0.8217 0.3968 0.7904
0.0938 0.3935 0.1500 0.2428 0.3947 0.3309 0.4299 0.8085 0.9493
0.5254 0.6714 0.5861 0.4424 0.6834 0.4243 0.8878 0.7551 0.3276
0.5303 0.7413 0.2621 0.6878 0.7040 0.2703 0.3912 0.3774 0.6713
0.8611 0.5201 0.0445 0.3592 0.4423 0.1971 0.7691 0.2160 0.4386
>> C=A*B
C =
1.4372 1.4996 0.7960 1.3779 1.6183 0.6385 1.7705 1.3605 1.6812
1.5484 1.6657 0.8419 1.5188 1.7849 0.7250 1.9207 1.5541 1.9734
1.5755 1.7888 1.0053 1.5573 1.9114 0.8555 2.1200 1.8162 2.0443
1.8700 1.7054 0.7593 1.6859 1.8605 0.5921 2.0316 1.2952 1.9852
1.1925 1.2892 0.5233 1.0441 1.2781 0.6230 1.4260 1.1911 1.5239
1.7873 1.7257 0.6737 1.6500 1.8091 0.6228 1.8500 1.2497 2.0004
1.6566 1.7241 0.8580 1.2840 1.7151 0.9249 2.1934 1.7681 1.8498
1.3942 1.4033 0.6158 1.0002 1.3626 0.7721 1.8188 1.4583 1.5942
0.9224 0.8985 0.5679 0.8773 1.0382 0.3677 1.2255 0.8932 1.0384
0.8602 1.0082 0.6461 0.8652 1.1167 0.5276 1.3329 1.2113 1.2512
```

```
>> [R,p]=rref(C)
R =
1.0000 0      0      0      0      -0.1962 0.7508 -0.6252 -0.8267
0      1.0000 0      0      0      0.6414 -1.4116 -0.9771 -1.7857
0      0      1.0000 0      0      -0.1448 -0.2807 -1.0964 -2.9899
0      0      0      1.0000 0      -1.0785 -2.3321 -3.1609 -2.6211
0      0      0      0      1.0000 0.9639 3.8590 5.5319 7.1301
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
p =
   1      2      3      4      5
>> D=C'
D =
1.4372 1.5484 1.5755 1.8700 1.1925 1.7873 1.6566 1.3942 0.9224 0.8602
1.4996 1.6657 1.7888 1.7054 1.2892 1.7257 1.7241 1.4033 0.8985 1.0082
0.7960 0.8419 1.0053 0.7593 0.5233 0.6737 0.8580 0.6158 0.5679 0.6461
1.3779 1.5188 1.5573 1.6859 1.0441 1.6500 1.2840 1.0002 0.8773 0.8652
1.6183 1.7849 1.9114 1.8605 1.2781 1.8091 1.7151 1.3626 1.0382 1.1167
0.6385 0.7250 0.8555 0.5921 0.6230 0.6228 0.9249 0.7721 0.3677 0.5276
1.7705 1.9207 2.1200 2.0316 1.4260 1.8500 2.1934 1.8188 1.2255 1.3329
1.3605 1.5541 1.8162 1.2952 1.1911 1.2497 1.7681 1.4583 0.8932 1.2113
1.6812 1.9734 2.0443 1.9852 1.5239 2.0004 1.8498 1.5942 1.0384 1.2512
>> [R,p]=rref(D)
R =
1.0000 0      0      0      0      3.2203 -2.3006 -4.4121 -2.3025 -3.9797
0      1.0000 0      0      0      1.8503 -4.8436 -4.6059 -1.1824 -0.8117
0      0      1.0000 0      0      -3.1623 4.5172 5.2312 2.6267 3.6533
0      0      0      1.0000 0      -0.8021 1.9536 2.6193 1.4479 1.4434
0      0      0      0      1.0000 0.6509 1.4196 1.4484 -0.6573 -0.5184
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0      0
p =
   1      2      3      4      5
```

From the out put above one can easily read bases for the four subspaces of $AB$:

$C(AB)$=row space of $D = (AB)^T$:
$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 3.2203 & -2.3006 & -4.4121 & -2.3025 & -3.9797\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 1.8503 & -4.8436 & -4.6059 & -1.1824 & -0.8117\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 1 & 0 & 0 & -3.1623 & 4.5172 & 5.2312 & 2.6267 & 3.6533\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 0 & 1 & 0 & -0.8021 & 1.9536 & 2.6193 & 1.4479 & 1.4434\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0.6509 & 1.4196 & 1.4484 & -0.6573 & -0.5184\end{pmatrix}$

$C((AB)^T)$=row space of $C = AB$:
$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & -0.1962 & 0.7508 & -0.6252 & -0.8267\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0.6414 & -1.4116 & -0.9771 & -1.7857\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 1 & 0 & 0 & -0.1448 & -0.2807 & -1.0964 & -2.9899\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 0 & 1 & 0 & -1.0785 & -2.3321 & -3.1609 & -2.6211\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0.9639 & 3.8590 & 5.5319 & 7.1301\end{pmatrix}$

$N(AB)$=nullspace of $C = AB$:
$\begin{pmatrix}-0.1962 & 0.6414 & -0.1448 & -1.0785 & 0.9639 & 1 & 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}0.7508 & -1.4116 & -0.2807 & -2.3321 & 3.8590 & 0 & 1 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}-0.6252 & -0.9771 & -1.0964 & -3.1609 & 5.5319 & 0 & 0 & 1 & 0\end{pmatrix}$
$\begin{pmatrix}-0.8267 & -1.7857 & -2.9899 & -2.6211 & 7.1301 & 0 & 0 & 0 & 1\end{pmatrix}$

$N((AB)^T)$=nullspace of $D = (AB)^T$:
$\begin{pmatrix}3.2203 & 1.8503 & -3.1623 & -0.8021 & 0.6509 & 1 & 0 & 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}-2.3006 & -4.8436 & 4.5172 & 1.9536 & 1.4196 & 0 & 1 & 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}-4.4121 & -4.6059 & 5.2312 & 2.6193 & 1.4484 & 0 & 0 & 1 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}-2.3025 & -1.1824 & 2.6267 & 1.4479 & -0.6573 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$
$\begin{pmatrix}-3.9797 & -0.8117 & 3.6533 & 1.4434 & -0.5184 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$

(b) The random matrices you constructed almost certainly have either full row or full column rank. Why?

$\boxed{\text{Solution}}$ For the matrix A not to have full column rank, some of the 5 columns would have to be linearly dependent, i.e. they would have to lie within a 4-dimensional subspace of $\mathbb{R}^{10}$. For random vectors to all lie within a lower-dimensional subspace is extremely unlikely. (For example, consider the probability that two random vectors in $\mathbb{R}^2$ are parallel. The odds of this are infinitesimal, i.e. zero, although strictly speaking on a computer there is some tiny finite chance because of the finite precision.) Similarly for B not to have full row rank.

(c) Suppose you were inventing homework problems for your classmates, and wanted to come up with random $7 \times 8$ matrix that is only rank 5...but it shouldn't be "obviously" rank 5 (i.e. it should be hard to tell just by looking at the matrix that the rows and columns are linearly dependent). Figure out a way to do this in Matlab, come up with a random $7 \times 8$ rank-5 matrix, check that it is rank 5, and with the help of the rref command find a basis for its four fundamental subspaces. (Hint: use a combination of random rank-5 matrices.)

$\boxed{\text{Solution}}$ We may first construct a random $7 \times 5$ matrix, then construct a random $5 \times 8$ matrix. Then their product will gives a random $7 \times 8$ rank-5 matrix. The commands and outputs are

```
>> A=rand(7,5);B=rand(5,8);C=A*B;[R,p]=rref(C),[R,p]=rref(C')
R =
1.0000 0      0      0      0      -0.3442 0.2002 0.4473
0      1.0000 0      0      0      4.8768 0.3169 1.4084
0      0      1.0000 0      0      4.4832 1.2324 1.0721
0      0      0      1.0000 0      -4.0481 -0.1616 -1.4489
0      0      0      0      1.0000 -2.0246 -0.5874 -0.5595
0      0      0      0      0      0      0      0
0      0      0      0      0      0      0      0
p =
    1      2      3      4      5
R =
1.0000 0      0      0      0      2.8588 1.0423
0      1.0000 0      0      0      0.0384 0.3094
0      0      1.0000 0      0      -0.1156 0.3103
0      0      0      1.0000 0      -2.3778 -1.1207
0      0      0      0      1.0000 0.1728 0.6989
0      0      0      0      0      0      0
0      0      0      0      0      0      0
0      0      0      0      0      0      0
p =
    1      2      3      4      5
```

It is of rank 5, and one can easily read bases of the four subspaces, as in part (a).

(One can also take the sum of 5 random rank-1 $7 \times 8$ matrices. To construct a random rank-1 matrix $7 \times 8$ matrix, one may first construct a random $7 \times 1$ vector and another random $1 \times 8$ vector, their matrix product is obviously of rank 1.)