18.06 Problem Set 10 - Solutions

Due **Thursday, 29 November** 2007 at 4 pm in 2-106.

Problem 1: (15=5+5+5) Take any matrix A of the form $A = B^H C B$, where B has full column rank and C is Hermitian and positive-definite.

(a) Show that A is Hermitian.

Solution For any two matrices A_1 and A_2, we have $(A_1 A_2)^T = A_2^T A_1^T$ and $\overline{A_1 A_2} = \bar{A}_1 \bar{A}_2$. Combine them, we have $(A_1 A_2)^H = A_2^H A_1^H$. Similarly, we have for any A, $(A^H)^H = A$.

Now back to the problem. Since C is Hermitian, $C^H = C$. So

$$A^H = (B^H C B)^H = B^H C^H (B^H)^H = B^H C B = A,$$

that is, A is Hermitian.

(b) Show that A is positive-definite by showing that $\mathbf{x} \cdot (A\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ (hint: very similar to how we showed that $B^H B$ is positive-definite, in class).

Solution Suppose $\mathbf{x} \neq \mathbf{0}$. Since B has full column rank, its nullspace only contains $\mathbf{0}$. So $B\mathbf{x} \neq \mathbf{0}$. So we have

$$\mathbf{x} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (B^H C B\mathbf{x}) = (B\mathbf{x}) \cdot C(B\mathbf{x}) > 0,$$

the last inequality comes from the fact that C is a positive definite matrix.

(c) Show that $A = D^H D$ for some D with full column rank. (Hint: use $\sqrt{C}$ as defined in an earlier problem set.)

Solution We first recall the definition of $\sqrt{C}$. Since C is Hermitian, it can be decomposed to $C = Q^H \Lambda Q$, where Q is unitary and Λ is diagonal whose diagonal entries are eigenvalues of C. Since C is positive definite, the diagonal entries of Λ are all positive, thus the square root matrix $\sqrt{C} = Q^H \sqrt{\Lambda} Q$ is well-defined, as we have seen in the earlier problem set. Notice that in particular we have $\sqrt{\Lambda}^H = \sqrt{\Lambda}$, thus $\Lambda = (\sqrt{\Lambda})^2 = \sqrt{\Lambda}^H \sqrt{\Lambda}$.

Now we have

$$A = B^H C B = B^H Q^H \Lambda Q B = B^H Q^H \sqrt{\Lambda}^H \sqrt{\Lambda} Q B.$$

If we denote $D = \sqrt{\Lambda} Q B$, we get immediately $A = D^H D$. Finally since both $\sqrt{\Lambda}$ and Q are nonsingular, and B is of full column rank, we see that D is of full column rank.

Problem 2: (25=4+5+5+5+3+3) Consider Poisson's equation $d^2f/dx^2 = g(x)$ [$g(x)$ is given and you want to find $f(x)$]. In lecture, we studied this for the case where f (and g) belongs to the space of real functions on $x \in [0,1]$ with $f(0) = f(1) = 0$: we solved it by expanding f and g in Fourier sine series and then inverting each eigenvalue. Now, you should see what happens in the space of functions with zero slope at the boundaries [$f'(0) = f'(1) = 0$], where the eigenfunctions of d^2/dx^2 gave the Fourier cosine series.

(a) What is the null space of d^2/dx^2? (Note that you should only consider functions in the vector space, i.e. with zero slope at $x = 0$ and $x = 1$.)

Solution Suppose

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = 0,$$

then $\frac{df}{dx} = a$ is constant. It follows that

$$f(x) = ax + b$$

is some linear function. Now since $f'(0) = f'(1) = 0$ and $f'(x) = a$, we get $a = 0$. So $f(x) = b$ is constant function. So the nullspace of d^2/dx^2 consists all the constant functions. (This is a one dimensional subspace.)

(b) What is the column space of d^2/dx^2, in terms of the Fourier cosine series? That is, if $d^2f/dx^2 = g(x)$, and you write out the cosine series of $g(x)$, what are the possible coefficients? (Hint: start with the cosine series of $f(x)$, and see what happens to it when you take the second derivative—what possible right-hand-sides can you get?)

Solution Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

Then

$$\frac{df}{dx} = -\sum_{n=1}^{\infty} n\pi a_n \sin(n\pi x)$$

and thus

$$\frac{d^2f}{dx^2} = -\sum_{n=1}^{\infty} n^2\pi^2 a_n \cos n\pi x.$$

So the column space of d^2/dx^2 consists all functions whose Fourier cosine series has vanishing first (0^{th} order) coefficient a_0, which by definition is $2\int_0^1 f(t)dt$. We conclude that the column space consists of functions f(x) with zero integral, i.e. functions whose average value is 0.

(c) Suppose that $g(x)$ is the function $g(x) = 1$ for $x < 1/2$ and $g(x) = -1$ for $x \geq 1/2$. Find the cosine series of $g(x)$, using the cosine-series formulas:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$
$$a_n = 2\int_0^1 g(x)\,\cos(n\pi x)\,dx.$$

[The $n = 0$ term has a $1/2$ factor in the first formula so that the second formula works for all n. The reason for the difference is just a matter of normalization: $\|\cos(0\pi x)\|^2 = 1$, but $\|\cos(n\pi x)\|^2 = 1/2$ for $n > 0$.] Hint: you should find that $a_n = 0$ for even n.

Solution We have

$$a_0 = 2\int_0^1 g(x)\,dx = 2\int_0^{1/2} dx + 2\int_{1/2}^1 (-1)\,dx = 0,$$

and for $n \geq 1$,

$$\begin{aligned} a_n &= 2\int_0^1 g(x)\,\cos(n\pi x)\,dx \\ &= 2\int_0^{1/2} \cos(n\pi x)\,dx - 2\int_{1/2}^1 \cos(n\pi x)\,dx \\ &= \frac{2}{n\pi}\sin(n\pi x)\Big|_0^{1/2} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{1/2}^1 \\ &= \frac{4}{n\pi}\sin(n\pi/2). \end{aligned}$$

Notice that $\sin(n\pi/2)$ equals 0 when n is even, equals 1 when n is of the form $4k+1$, and equals -1 when n is of the form $4k+3$. We finally find the cosine series of g,

$$\begin{aligned} g(x) &= \sum_{k=1}^{\infty}\left(\frac{4}{(4k+1)\pi}\cos[(4k+1)\pi x] - \frac{4}{(4k+3)\pi}\cos[(4k+3)\pi x]\right) \\ &= \frac{4}{\pi}\cos(\pi x) - \frac{4}{3\pi}\cos(3\pi x) + \frac{4}{5\pi}\cos(5\pi x) - \frac{4}{7\pi}\cos(7\pi x) + \cdots \end{aligned}$$

(d) Verify that $g(x)$ from (c) is in the column space from (b). Using your answer from (c), find the cosine series for $f(x)$ to satisfy Poisson's equation. $f(x)$ should be the sum of a particular solution plus an arbitrary nullspace solution, using your answer to (a).

Solution Since $a_0 = 0$, $g(x)$ lies in the column space of the operator d^2/dx^2.

To find a particular solution $f(x)$, we suppose $f(x)$ has the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

We have seen above that

$$\frac{d^2 f}{dx^2} = -\sum_{n=1}^{\infty} n^2\pi^2 a_n \cos n\pi x.$$

Compare its coefficients with the coefficients of the cosine series of g above, we conclude that $a_n = 0$ for even $n \geq 2$, $a_n = -\frac{4}{n^3\pi^3}$ for odd n of the form $4k+1$, and $a_n = \frac{4}{n^3\pi^3}$ for odd n of the form $4k+3$. There is no restriction on a_0, so it can be arbitrary constant, which corresponds to arbitrary nullspace solution. (The Fourier series method is easy to solve ODE, since inverting d^2/dx^2 corresponds to simply multiplying each eigenfunction by the inverse of the eigenvalues.)

(e) In Matlab, plot the first four nonzero terms of your $g(x)$ cosine series, and then plot the first 8 nonzero terms—verify that the cosine series is converging to $g(x)$ (except right at the point of the discontinuity). For example, if you put the coefficients in the variables `a0`, `a1`, and so on (e.g. `a0 = 1.234/pi`), then you can plot the first four terms of the Fourier cosine series with the command:

```
fplot(@(x) a0/2 + a1*cos(pi*x) + a2*cos(2*pi*x) + a3*cos(3*pi*x), [0,1])
```

Solution The Inputs are

```
>>  fplot(@(x) 4/pi*cos(pi*x) -4/(3*pi)*cos(3*pi*x)
    + 4/(5*pi)*cos(5*pi*x) - 4/(7*pi)*cos(7*pi*x), [0,1])

>>  fplot(@(x) 4/pi*cos(pi*x) -4/(3*pi)*cos(3*pi*x)
    + 4/(5*pi)*cos(5*pi*x) - 4/(7*pi)*cos(7*pi*x)
    + 4/(9*pi)*cos(9*pi*x) - 4/(11*pi)*cos(11*pi*x)
    + 4/(13*pi)*cos(13*pi*x) - 4/(15*pi)*cos(15*pi*x), [0,1])
```

Outputs

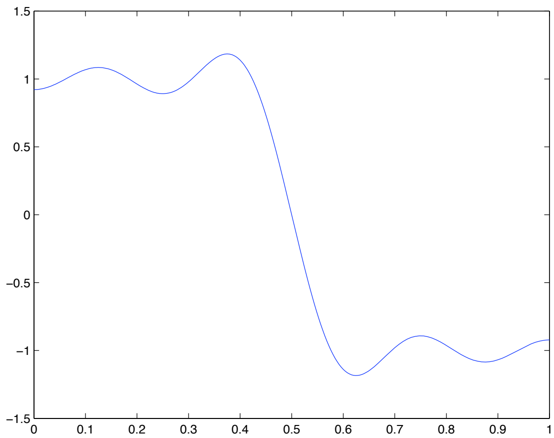

Figure 1: 4 nonzero terms of g

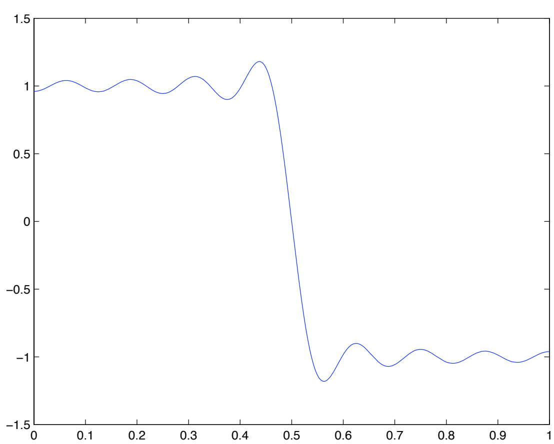

Figure 2: 8 nonzero terms of g

(f) As in (e), but plot the first 4 and 8 non-zero terms of your solution $f(x)$ from (d) [just pick some value for the nullspace part of the solution]. Which series converges faster, the one for f or the one for g?

Solution We pick the arbitrary constant a_0 to be zero. The codes:

```
>>  fplot(@(x) -4/(pi^3)*cos(pi*x) +4/(3^3*pi^3)*cos(3*pi*x)
    - 4/(5^3*pi^3)*cos(5*pi*x) + 4/(7^3*pi^3)*cos(7*pi*x), [0,1])

>>  fplot(@(x) -4/(pi^3)*cos(pi*x) +4/(3^3*pi^3)*cos(3*pi*x)
    - 4/(5^3*pi^3)*cos(5*pi*x) + 4/(7^3*pi^3)*cos(7*pi*x), [0,1])
    - 4/(9^3*pi^3)*cos(9*pi*x) + 4/(11^3*pi^3)*cos(11*pi*x)
    - 4/(13^3*pi^3)*cos(13*pi*x) + 4/(15^3*pi^3)*cos(15*pi*x), [0,1])
```

Outputs

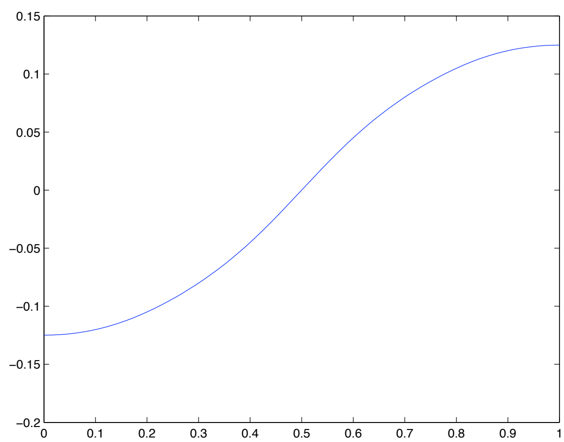

Figure 3: 4 nonzero terms of f

It turns out that the cosine series for f converges much faster. This is true in general, since the denominator of each term in the cosine series of f is much bigger than the corresponding term of g. (There is an extra n^2 term)

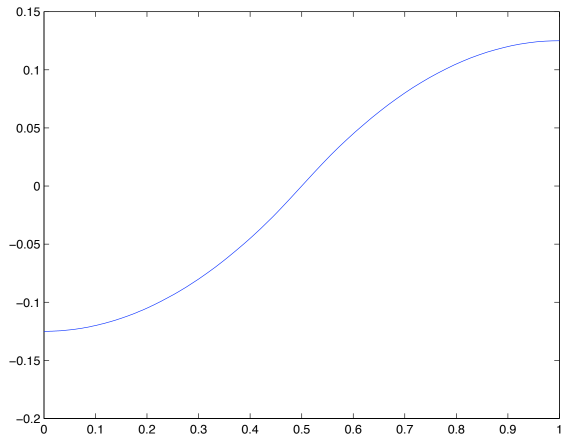

Figure 4: 8 nonzero terms of f

Problem 3: (18=13+5) (1) Follow the steps in problem 13 on page 350 to show that A^T is always similar to A.

Solution

▶ Step 1: J_i is similar to J_i^T.

Recall that any Jordan block is of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

Take M_i to be the anti-diagonal matrix whose anti-diagonal elements are 1, and whose size is the same as J_i, i.e.

$$M_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Then one can easily check that $M_i^{-1} = M_i$. Notice that multiply to the left by M_i just changes the rows from $1, 2, \cdots, n$ to $n, n-1, \cdots, 1$, and multiply to the right by M_i just changes the columns from $1, 2, \cdots, n$ to $n, n-1, \cdots, 1$. So it follows

$$M_i^{-1} J_i M_i = M_i J_i M_i = J_i^T.$$

▶ Step 2: Now given the Jordan form

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & J_r \end{pmatrix},$$

we take

$$M = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & M_r \end{pmatrix},$$

it follows from part one that $M^{-1} J M = J^T$.

▶ Step 3: Suppose $A = PJP^{-1}$, where J is Jordan form above. Take $Q = MP^T$, then

$$A^T = (P^{-1})^T J^T P^T = (P^T)^{-1} M^{-1} J M P^T = Q^{-1} J Q,$$

so A^T is similar to A.

(2) Is A^H always similar to A? Justify your conclusion.

Solution No.

Since the eigenvalues of A^T are the same eigenvalues of A, we see that the eigenvalues of A^H are the conjugate of the eigenvalues of A. So in general A^H has different eigenvalues of A, and thus they are not similar.

Problem 4: (16=4+4+4+4) Let $A = \mathbf{u}\mathbf{v}^T$ be any rank-1 matrix.

(1) What is the dimension of $N(A)$? What is $C(A)$?

Solution Since A is rank-1 matrix, the dimension of $N(A)$ is $n-1$.

(For any two matrices A and B, the column space of AB always lies in the column space of A.) Since the column space of A is one dimensional, and is contained in the line spanned by $\mathbf{u}$, we see that the column space of A is exactly the line spanned by $\mathbf{u}$.

(2) Find all eigenvalues of A, assuming that $\mathbf{u}$ and $\mathbf{v}$ both have n components so that A is square.

Solution Since $N(A)$ is $n-1$ dimensional, 0 is an eigenvalue of A with multiplicity $n-1$.

Since $C(A)$ is one dimensional and contains $\mathbf{u}$, the vector $\mathbf{u}$ must be an eigenvector. Now

$$A\mathbf{u} = (\mathbf{u}\mathbf{v}^T)\mathbf{u} = \mathbf{u}(\mathbf{v}^T\mathbf{u}) = (\mathbf{v}^T\mathbf{u})\mathbf{u},$$

so the corresponding eigenvalue (the last one) is $\mathbf{v}^T\mathbf{u}$. (However, if $\mathbf{v}^T\mathbf{u} = 0$, then all eigenvalues are 0, AND there are only $n-1$ eigenvectors, in which case the matrix is not diagonalizable.)

(3) What are the singular values of A? What is an SVD for A?

Solution Since $AA^T = \mathbf{u}\mathbf{v}^T\mathbf{v}\mathbf{u}^T = (\mathbf{v}^T\mathbf{v})\mathbf{u}\mathbf{u}^T$, so the eigenvalues of AA^T are 0 (of multiplicity $n-1$) and $(\mathbf{v}^T\mathbf{v})(\mathbf{u}^T\mathbf{u})$ (which is nonzero). Thus the singular values of A is $0, \cdots, 0, \sqrt{(\mathbf{v}^T\mathbf{v})(\mathbf{u}^T\mathbf{u})}$.

Now SVD for A is $U\Sigma V$, where U is unitary whose first $n-1$ columns are an orthonormal basis of $N(A)$ (= the hyperplane whose normal vector is $\mathbf{u}$), and the last column is $\hat{\mathbf{u}}$, the normalization of $\mathbf{u}$, and V has the same description, replacing $\mathbf{u}$ by $\mathbf{v}$.

(4) Construct a rank-1 matrix A so that $A(1,0,1)^T = (2,1)^T$.

Solution A must be a 2×3 matrix. Thus $A = \mathbf{u}\mathbf{v}^T$, with $\mathbf{u}$ a 2-vector, and $\mathbf{v}$ a 3-vector. Since $(2,1)^T$ lies in the column space which is one dimensional, we see from above that it is eigenvector which is a multiple of $\mathbf{u}$. We can take $\mathbf{u} = (2,1)^T$. Now from $A(1,0,1)^T = (2,1)^T$ we get $\mathbf{v}^T(1,0,1)^T = 1$. We can take, for example, $\mathbf{v} = (1,0,0)^T$. Thus we may take $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(The general form of $\mathbf{v}$ is $(a, b, 1-a)^T$, and the general A is $A = \begin{pmatrix} 2a & 2b & 2-2a \\ a & b & 1-a \end{pmatrix}$).

Problem 5: (16=4+4+4+4) (1) Find the SVD for $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution We have $A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, whose eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$, and corresponding eigenvectors, after normalization, are $\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$. It follows that

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} A\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \quad \mathbf{u}_2 = A\mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}.$$

The vector $\mathbf{u}_3$ is the unit vector in the nullspace of A^T, which is $\mathbf{u}_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$. So the SVD for A is

$$A = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(2) Find the pseudoinverse B of A.

Solution The pseudoinverse B of A is

$$B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$
$$= \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}.$$

(3) Compute AB, BA, ABA, and BAB.

Solution Just straightforward computation:

$$AB = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $ABA = A(BA) = AI = A$, $BAB = (BA)B = IB = B$.

(4) What are ABA and BAB for a general matrix A and its pseudoinverse B, given the definitions of the SVD and pseudoinverse?

Solution We will always have $ABA = A$ and $BAB = B$. In fact, we have

$$\Sigma^+\Sigma = \begin{pmatrix} 1/\sigma_1 & & & & \\ & \ddots & & & \\ & & 1/\sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix},$$

it follows $\Sigma\Sigma^+\Sigma = \Sigma$ and $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$. Thus

$$ABA = U\Sigma V^H V\Sigma^+ U^H U\Sigma V^H = U\Sigma\Sigma^+\Sigma V^H = U\Sigma V^H = A$$

and

$$ABA = V\Sigma^+ U^H U\Sigma V^H V\Sigma^+ U^H = V\Sigma^+\Sigma\Sigma^+ U^H = V\Sigma^+ U^H = B.$$

Problem 6: (10=5+5) Given the SVD $A = U\Sigma V^H$.

(1) What is the SVD of A^H and A^{-1} (assuming A is invertible)?

Solution For A^H we have $A^H = V\Sigma^H U^H = V\Sigma U^H$.

Suppose A is invertible, then it is square matrix, and Σ is invertible diagonal matrix. So the SVD of A^{-1} is $A^{-1} = V\Sigma^{-1}U^H$.

(2) If the QR decomposition of A is $A = QR$, what is the SVD for R?

Solution If A is a square matrix, then $A = QR = U\Sigma V^H$, so it follows $R = (Q^H U)\Sigma V^H$, which is the SVD for R.

If A is an $m \times n$ non-square with full column rank (thus $m \geq n$), then $A = QR$ is also defined to be the result of Gram-Schmidt, in which case Q is $m \times n$ matrix whose columns are orthonormal vectors, and R is $n \times n$ and invertible. In this case, given the SVD $A = U\Sigma V$, we may take only the first n columns of U, and the upper-left $n \times n$ corner of Σ, corresponding to a basis for $C(A)$ only, then apply above arguments.