Practice 18.06 Final Questions with Solutions

17th December 2007

Notes on the practice questions

The final exam will be on Thursday, Dec. 20, from 9am to 12noon at the Johnson Track, and will most likely consist of 8–12 questions. The practice problems below mostly concentrate on the material from exams 1–2, since you already have practice problems for exam 3. The real final will have plenty of eigenproblems!

These questions are intended to give you a flavor of how I want you to be able to think about the material, and the flavor of possible questions I might ask. Obviously, these questions do *not* exhaust *all* the material that we covered this term, so you should of course still study your lecture notes and previous exams, and review your homework.

Solutions for these practice problems should be posted on the 18.06 web site by 12/15.

List of potential topics:

Material from exams 1, 2, and 3, and the problem sets (and lectures) up to that point.

Definitely not on final: finite-difference approximations, sparse matrices and iterative methods, non-diagonalizable matrices, generalized eigenvectors, principal components analysis, choosing a basis to convert a linear operator into a matrix, numerical linear algebra and error analysis.

Key ideas:

- The four subspaces of a matrix A and their relationships to one another and the solutions of $A\mathbf{x} = \mathbf{b}$.
- Gaussian elimination $A \to U \to R$ and backsubstitution. Elimination = invertible row operations = multiplying A on the *left* by an invertible matrix. Multiplying on left by an invertible matrix preserves $N(A)$, and hence we can use elimination to find the nullspace. Also, it thus preserves $C(A^H) = N(A)^\perp$. Also, it thus preserves the linear independence of the columns and hence the pivot columns in A are a basis for $C(A)$. Conversely, invertible *column* operations = multiplying A on the *right* by an invertible matrix, hence preserving $N(A^H)$ and $C(A)$.
- Solution of $A\mathbf{x} = \mathbf{b}$ when A is *not* invertible: exactly solvable if $\mathbf{b} \in C(A)$, particular solutions, not unique if $N(A) \neq \{\mathbf{0}\}$. Least-squares solution $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$ if A full column rank, equivalence to minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$, relationship $A\hat{\mathbf{x}} = P_A\mathbf{b}$ to projection matrix $P_A = A(A^H A)^{-1}A^H$ onto $C(A)$. The fact that $\operatorname{rank}(A^H A) = \operatorname{rank} A = \operatorname{rank} A^H$, hence $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$ is always solvable for *any* A, and $A^H A$ is always invertible (and positive-definite) if A has full column rank.
- Vector spaces and subspaces. Dot products, transposes/adjoints, orthogonal complements. Linear independence, bases, and orthonormal bases. Gram-Schmidt and QR factorization.

Unitary matrices $Q^H = Q^{-1}$, which preserve dot products $\mathbf{x} \cdot \mathbf{y} = (Q\mathbf{x}) \cdot (Q\mathbf{y})$ and lengths $\|\mathbf{x}\| = \|Q\mathbf{x}\|$.

- Determinants of square matrices: properties, invariance under elimination (row swaps flip sign), = product of eigenvalues. The trace of a matrix = sum of eigenvalues. $\det AB = (\det A)(\det B)$, $\operatorname{trace} AB = \operatorname{trace} BA$, $\operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B$.

- For an eigenvector, any complicated matrix or operator acts just like a number λ, and we can do anything we want (inversion, powers, exponentials...) using that number. To act on an arbitrary vector, we expand that vector in the eigenvectors (in the usual case where the eigenvectors form a basis), and then treat each eigenvector individually. Finding eigenvectors and eigenvalues is complicated, though, so we try to infer as much as we can about their properties from the structure of the matrix/operator (Hermitian, Markov, etcetera). SVD as generalization of eigenvectors/eigenvalues.

Problem 1

Suppose that A is some real matrix with full column rank, and we do Gram-Schmidt on it to get an the following orthonormal basis for $C(A)$: $\mathbf{q}_1 = (1,0,1,0,-1)^T/\sqrt{3}$, $\mathbf{q}_2 = (1,2,-1,0,0)^T/\sqrt{6}$, $\mathbf{q}_3 = (-2,1,0,0,2)^T/\sqrt{9}$.

(a) Suppose we form the matrix B whose columns are $\sqrt{3}\mathbf{q}_1$, $\sqrt{6}\mathbf{q}_2$, and $\sqrt{9}\mathbf{q}_3$. Which of the four subspaces, if any, are guaranteed to be the same for A and B?

(b) Find a basis for the left nullspace $N(A^T)$.

(c) Find a basis for the row space $C(A^T)$.

$\boxed{\text{Solution:}}$

(a) All four subspaces for A and B are the same. Obviously B has the same column space as A, i.e. $C(A) = C(B)$. It follows that they have the same left nullspace, $N(A^T) = N(B^T)$, which are just the orthogonal complement of column space. Since both A and B has full column rank, $N(A) = N(B) = \{0\}$. It follows that the row spaces $C(A^T) = C(B^T)$.

(b) As we have observed above, $N(A^T) = N(B^T)$, where $B = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. To find a basis for B^T, we do Gauss elimination:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & -2 & 0 & 1 \\ 0 & 0 & 3 & 0 & -1/2 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -5/6 \\ 0 & 2 & 0 & 0 & 2/3 \\ 0 & 0 & 3 & 0 & -1/2 \end{pmatrix},$$

So we may take two vectors $(0,0,0,1,0)^T$ and $(5,-2,1,0,6)^T$ as a basis of $N(A^T)$.

(c) A is a 5×3 matrix with full column rank, i.e. $\operatorname{rank}(A) = 3$. So the row space have dimension $\dim(C(A^T)) = 3$. However, $C(A^T)$ lies in $\mathbb{R}^3$. So $C(A^T) = \mathbb{R}^3$. We may take the vectors $(1,0,0)^T, (0,1,0)^T, (0,0,1)^T$ as a basis.

Problem 2

Perverse physicist Pat proposes a permutation: Pat permutes the columns of some matrix A by some random sequence of column swaps, resulting in a new matrix B.

(a) If you were given B (only), which of the four subspaces of A (if any) could you find? (i.e. which subspaces are preserved by column swaps?)

(b) Suppose $B = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $\mathbf{b} = (0, 1, 2)^T$. Check whether $A\mathbf{x} = \mathbf{b}$ is solvable, and if so whether the solution $\mathbf{x}$ is unique.

(c) Suppose you were given B and it had full column rank. You are also given $\mathbf{b}$. Give an explicit formula (in terms of B and $\mathbf{b}$ only) for the *minimum* possible *value* of $\|A\mathbf{x} - \mathbf{b}\|^2$.

Solution:

(a) Only column swaps are performed, so the column space is not changed. As a consequence, the left nullspace is also not changed. So given B, we can find $C(A)$ and $N(A^T)$.

(b) Since $C(A) = C(B)$, the solvability of $A\mathbf{x} = \mathbf{b}$ is equivalent to the solvability of $B\mathbf{x} = \mathbf{b}$. We do Gauss elimination for the second equation:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & 5 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 4 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 10/3 & 2/3 \end{pmatrix}.$$

So the equation $B\mathbf{x} = \mathbf{b}$, and thus the equation $A\mathbf{x} = \mathbf{b}$, is solvable. Moreover, the solution $\mathbf{x}$ is unique.

(c) All the vectors $A\mathbf{x}$ are exactly the vectors in the column space $C(A) = C(B)$. Thus the minimum possible value of $\|A\mathbf{x} - \mathbf{b}\|^2$ is exactly the minimum possible value of $\|B\mathbf{x} - \mathbf{b}\|^2$. To find the latter one, we need the vector $B\mathbf{x}$ to be the vector in $C(B)$ which is closest to the vector $\mathbf{b}$. In other words, $B\mathbf{x}$ should be the projection of $\mathbf{b}$ on $C(B)$. Since B is of full column rank, the projection is $B\mathbf{x} = B(B^TB)^{-1}B^T\mathbf{b}$. So the minimum value we want is $\|B(B^TB)^{-1}B^T\mathbf{b} - \mathbf{b}\|^2$

Problem 3

Which of the following sets are vector spaces (under ordinary addition and multiplication by real numbers)?

(a) Given a vector $\mathbf{x} \in \mathbb{R}^n$, the set of all vectors $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \cdot \mathbf{y} = 3$.

(b) The set of all functions $f(x)$ whose integral $\int_{-\infty}^{\infty} f(x)dx$ is zero.

(c) Given a subspace $V \subseteq \mathbb{R}^n$ and an $m \times n$ matrix A, the set of all vectors $A\mathbf{x}$ for all $\mathbf{x} \in V$.

(d) Given a line $L \subseteq \mathbb{R}^n$ and an $m \times n$ matrix A, the set of all vectors $A\mathbf{x}$ for all $\mathbf{x} \in L$.

(e) The set of $n \times n$ Markov matrices.

(f) The set of eigenvectors with $|\lambda| < \frac{1}{2}$ of an $n \times n$ Markov matrix A.

Solution:

(a) This is not a vector space, since it doesn't contain the zero vector.

(b) This is a vector space.

It is a subset of the vector space consisting all functions. If f and g both has whole integral zero, $af + bg$ also has whole integral zero. So it is a vector subspace.

(c) This is a vector space.

It is just the column space of A, which is a vector subspace of $\mathbb{R}^m$.

(d) This is not a vector space.

A line in $\mathbb{R}^n$ doesn't have to pass the origin, so the set of all vectors $A\mathbf{x}$ need not contain the zero vector.

(e) This is not a vector space.

The zero matrix is not a Markov matrix.

(f) This is not a vector space in general.

If the Markov matrix A has more than one eigenvalue whose absolute value are less than $1/2$, then the corresponding set of eigenvectors is not a vector space, since the summation of two eigenvectors corresponding to different eigenvalues is not a eigenvector.

Problem 4

The *rows* of an $m \times n$ matrix A are linearly independent.

(a) Is $A\mathbf{x} = \mathbf{b}$ necessarily solvable?

(b) If $A\mathbf{x} = \mathbf{b}$ is solvable, is the solution necessarily unique?

(c) What are $N(A^H)$ and $C(A)$?

$\boxed{\text{Solution:}}$

(a) Yes. Since the rows of A are linearly independent, $\text{rank}(A) = m$. So the column space of A is an m-dimensional subspace of $\mathbb{R}^m$, i.e., is $\mathbb{R}^m$ itself. It follows that for any $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ is always solvable.

(b) No, the solution maybe not unique. Since $\text{rank}(A) = m$, the nullspace is $n-m$ dimensional. Thus the solution is not unique if $n > m$, and is unique if $n = m$. (It will never have that $n < m$, otherwise the rows are not linearly independent.)

(c) We have seen in part (a) that $C(A) = \mathbb{R}^m$. So $N(A^T) = \{0\}$. Since $N(A^H)$ consists those points whose conjugate lies in $N(A^T)$, we see $N(A^H) = \{0\}$.

Problem 5

Make up your own problem: give an example of a matrix A and a vector $\mathbf{b}$ such that the solutions of $A\mathbf{x} = \mathbf{b}$ form a line in $\mathbb{R}^3$, $\mathbf{b} \neq \mathbf{0}$, and all the entries of the matrix A are nonzero. Find all solutions $\mathbf{x}$.

$\boxed{\text{Solution:}}$

Such a matrix must be an $m \times 3$ matrix whose nullspace is one dimensional. In other words, the rank is $3 - 1 = 2$. We may take A to be an 2×3 matrix whose rows are linearly independent. As an example, we take

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find all solutions, we do elimination

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so the solutions are given by $\mathbf{x} = (1 - t, t, 0)^T$.

Problem 6

Clever Chris the chemist concocts a conundrum: in ordinary least-squares, you minimize $\|A\mathbf{x}-\mathbf{b}\|^2$ by solving $A^H A\hat{\mathbf{x}} = A^H\mathbf{b}$, but suppose instead that we wanted to minimize $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$ for some Hermitian positive-definite matrix C?

(a) Suppose $C = B^H B$ for some matrix B. Which of the following (if any) must be properties of B: (i) Hermitian, (ii) Markov, (iii) unitary, (iv) full column rank, (v) full row rank?

(b) In terms of B, A, and $\mathbf{b}$, write down an explicit formula for the $\mathbf{x}$ that minimizes $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$.

(c) Rewrite your answer from (b) purely in terms of C, A, and $\mathbf{b}$.

(d) Suppose that C was only positive semi-definite. Is there still a minimum value of $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$? Still a unique solution $\mathbf{x}$?

Solution:

(a) For $C = B^H B$ a Hermitian positive-definite matrix, B must be (iv) full column rank. To show this, we only need to notice that $\mathbf{x}\cdot C\mathbf{x} = \mathbf{x}\cdot B^H B\mathbf{x} = \|Bx\|^2$. Since C is positive definite, B has no nonzero nullspace. It follows that B is of full column rank.

B don't have to be Hermitian or Markov or unitary or full row rank. For example, we may take $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $C = B^H B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) We have

$$\begin{aligned}(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b}) &= (A\mathbf{x}-\mathbf{b})^H B^H B(A\mathbf{x}-\mathbf{b}) = (BA\mathbf{x}-B\mathbf{b})^H(BA\mathbf{x}-B\mathbf{b}) \\ &= \|BA\mathbf{x}-B\mathbf{b}\|^2.\end{aligned}$$

To minimize this, we need to solve the equation $(BA)^H(BA)\hat{x} = (BA)^H B\mathbf{b}$. Since B and A are of full column rank, BA is of full column rank. So $(BA)^H(BA)$ is invertible. The solution is given by $\hat{x} = (A^H B^H BA)^{-1}A^H B^H V\mathbf{b}$.

(c) Since $B^H B = C$, we have $\hat{x} = (A^H CA)^{-1}A^H C\mathbf{b}$.

(d) If C is only positive semi-definite, we still have $C = B^H B$ for some B, but B need not to be of full column rank. As above, we have

$$(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b}) = \|BA\mathbf{x}-B\mathbf{b}\|^2.$$

So the minimum value of $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$ still exists. However, since B may be not of full column rank, the matrix need not be invertible. The solution $\mathbf{x}$ need not be unique. For example, if we take C to be the zero matrix, any $\mathbf{x}$ will minimize $(A\mathbf{x}-\mathbf{b})^H C(A\mathbf{x}-\mathbf{b})$.

Problem 7

True or false (explain why if true, give a counter-example if false).

(a) For $n\times n$ real-symmetric matrices A and B, AB and BA always have the same eigenvalues. [Hint: what is $(AB)^T$?]

(b) For $n\times n$ matrices A and B with B invertible, AB and BA always have the same eigenvalues. [Hint: you can write $\det(AB-\lambda I) = \det((A-\lambda B^{-1})B)$. Alternative hint: think about similar matrices.]

(c) Two diagonalizable matrices A and B with the same eigenvalues and eigenvectors must be the same matrix.

(d) Two diagonalizable matrices A and B with the same eigenvalues must be the same matrix.

(e) For $n \times n$ matrices A and B with B invertible, AB and BA always have the same eigenvectors.

$\boxed{\text{Solution:}}$

(a) True.

Since A and B are real-symmetric matrices, $(AB)^T = B^T A^T = BA$. But $(AB)^T$ has the same eigenvalue as AB. So AB and BA have the same eigenvalues.

(b) True.

Since B is invertible, we have $AB = B^{-1}(BA)B$. So BA is similar to AB, and they must have the same eigenvalues.

(c) True.

Let Λ be the diagonal matrix consists of eigenvalues, and S be the matrix of eigenvectors (ordered as the eigenvalue matrix). Then we have $A = S^{-1}\Lambda S$ and also $B = S^{-1}\Lambda S$. So $A = B$.

(d) False.

For example, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are both diagonalizable, with the same eigenvalues, but they are different.

(e) False.

For example, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The eigenvectors of AB are $(0,1)^T$ and $(1,0)^T$, while the eigenvectors or BA are $(0,1)^T$ and $(1,1)^T$.

Problem 8

You are given the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(a) What is the sum of the eigenvalues of A?

(b) What is the product of the eigenvalues of A?

(c) What can you say, without computing them, about the eigenvalues of AA^T?

$\boxed{\text{Solution:}}$

(a) The sum of the eigenvalues of A equals trace of A. So the sum of the eigenvalues of A is $0 + 1 + 1 - 1 = 1$.

(b) The product of the eigenvalues of A equals the determinant of A. We do row transforms

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

where in the first step we add the other rows to the first row. It follows that the product of eigenvalues equals $\det(A) = -2$.

(c) AA^T is a real symmetric matrix, so its eigenvalues are all real and nonnegative. Since A is nonsingular, all eigenvalues of AA^T are positive. The product of all eigenvalues of AA^T are

$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 4$. The sum of all eigenvalues of AA^T is $\text{trace}(AA^T) = \sum_{i,j} a_{ij}^2 = 9$.

Problem 9

You are given the quadratic polynomial $f(x, y, z)$:

$$f(x, y, z) = 2x^2 - 2xy - 4xz + y^2 + 2yz + 3z^2 - 2x + 2z.$$

(a) Write $f(x, y, z)$ in the form $f(x, y, z) = \mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$ where $\mathbf{x} = (x, y, z)^T$, A is a real-symmetric matrix, and $\mathbf{b}$ is some constant vector.

(b) Find the point (x, y, z) where $f(x, y, z)$ is at an extremum.

(c) Is this point a minimum, maximum, or a saddle point of some kind?

Solution:

(a) We have

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$

(b) We compute the partial derivatives to find the extremum point:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 2y - 4z - 2 = 0 \\ \frac{\partial f}{\partial y} &= -2x + 2y + 2z = 0 \\ \frac{\partial f}{\partial x} &= -4x + 2y + 6z + 2 = 0 \end{aligned}$$

The equation is just $2A\mathbf{x} = \mathbf{b}$. The solution to the the equations above is $x = 1, y = 1, z = 0$. So the extreme point is $(1, 1, 0)$.

(c) We look for the pivots of A:

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since A is positive definite, the extreme point is a minimum.

Problem 10

Suppose A is some diagonalizable matrix. Consider the vector $\mathbf{y}(t) = e^{A^2 t}\mathbf{x}$ for some vector $\mathbf{x}$.

(a) If A is 3×3 with eigenvalues λ_1, λ_2, and λ_3 with eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, and $\mathbf{x} = \mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3$, what is $\mathbf{y}(t)$?

(b) If $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ for *every* vector $\mathbf{x}$, what does that tell you about the eigenvalues of A and of e^{A^2}?

(c) In terms of A, give a system of differential equations that $\mathbf{y}(t)$ satisfies, and the initial condition.

(d) In terms of A, give a linear recurrence relation that $\mathbf{y}(t)$ satisfies for $t = k\Delta t$ for integers k and some fixed Δt.

Solution:

(a) A^2 has eigenvalues λ_1^2, λ_2^2 and λ_3^2 with eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$, so $e^{A^2 t}$ has eigenvalues $e^{\lambda_1^2 t}, e^{\lambda_2^2 t}$ and $e^{\lambda_3^2 t}$ with the same eigenvectors. Thus

$$\mathbf{y}(t) = e^{A^2 t}\mathbf{x} = e^{A^2 t}(\mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3) = e^{\lambda_1^2 t}\mathbf{x}_1 + 3e^{\lambda_2^2 t}\mathbf{x}_2 + 4e^{\lambda_3^2 t}\mathbf{x}_3.$$

(b) If $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ for every vector $\mathbf{x}$, we must have $\operatorname{Re}\lambda_1^2 < 0, \operatorname{Re}\lambda_2^2 < 0$ and $\operatorname{Re}\lambda_3^2 < 0$. The eigenvalues $e^{\lambda_1^2}, e^{\lambda_2^2}$ and $e^{\lambda_3^2}$ all sastisfies $|e^{\lambda^2}| < 1$.

(c) $\mathbf{y}(t)$ satisfies the system $\mathbf{y}' = A^2\mathbf{y}$. The initial condition is $\mathbf{y}(0) = \mathbf{x}$.

(d) Denote by $\mathbf{y}_k = \mathbf{y}(k\Delta t)$. Then equation $\mathbf{y}(t) = e^{A^2 t}\mathbf{x}$ gives the recurrence relation $\mathbf{y}_{k+1} = e^{A^2\Delta t}\mathbf{y}_k$ with initial condition $\mathbf{y}_0 = \mathbf{x}$.

Problem 11

Suppose A is an $m \times n$ matrix with full *row* rank. Which of the following equations *always* have a solution (possibly non-unique) for any $\mathbf{b}$?

(a) $A\mathbf{x} = \mathbf{b}$

(b) $A^H\mathbf{x} = \mathbf{b}$

(c) $A^H A\mathbf{x} = \mathbf{b}$

(d) $AA^H\mathbf{x} = \mathbf{b}$

(e) $A^H A\mathbf{x} = A^H\mathbf{b}$

(f) $AA^H\mathbf{x} = A\mathbf{b}$

Solution:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, as explained in problem 4.

(b) The equation $A^H\mathbf{x} = \mathbf{b}$ may has no solution, since the column space of A^H is an m-dimensional subspace in the whole space $\mathbb{R}^n$.

(c) The equation $A^H A\mathbf{x} = \mathbf{b}$ may has no solution, since the $C(A^H A) = C(A^H)$ is an m-dimensional subspace in $\mathbb{R}^n$. (That $C(A^H A) = C(A^H)$ comes from the fact $N(A^H A) = N(A)$ which we proved in class.)

(d) The equation $AA^H\mathbf{x} = \mathbf{b}$ will always has a unique solution, since AA^H is an $m\times m$ matrix whose rank is m, i.e. AA^H is invertible.

(e) The equation $A^H A\mathbf{x} = A^H\mathbf{b}$ will always has a solution. In fact, any solution to $A\mathbf{x} = \mathbf{b}$ (from part (a)) is always a solution to $A^H A\mathbf{x} = A^H\mathbf{b}$.

(f) The equation $AA^H\mathbf{x} = A\mathbf{b}$ will always has a unique solution, since AA^H is invertible, as explained in part (d).

Additional Practice Problems

Be sure to look at:

(i) Exams 1, 2, and 3.

(ii) The practice problems for exam 3. (The above problems mostly cover non-eigenvalue stuff.)

(iii) Ideally, also review your homework problems.