18.06 Problem Set 9

Due **TUESDAY**, Nov. 22, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 11/13*

Do Problem #10 from section 6.3 in your book. (For (a), just explain in your own words: Why is u's length constant? And why is that constant the length of $u(0)$?)

Solution 1

(a) The value of $\|u(t)\|^2$ (hence the length of $u(t)$) doesn't change, because its derivative is zero. Verify this: $2u_1u_1' + 2u_2u_2' + 2u_3u_3' = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = 0$. So its value is the same as its value at time zero, which is $\|u(0)\|^2$.
(b)

$$\begin{aligned} Q &= \exp(At) \\ &= I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots \\ Q^{\mathrm{T}} &= (\exp(At))^{\mathrm{T}} \\ &= I^{\mathrm{T}} + A^{\mathrm{T}}t + (A^{\mathrm{T}})^2\frac{t^2}{2!} + (A^{\mathrm{T}})^3\frac{t^3}{3!} + \dots \\ &= I + (-A)t + (-A)^2\frac{t^2}{2!} + (-A)^3\frac{t^3}{3!} + \dots \\ &= \exp(-At) \end{aligned}$$

Problem 2 *Wednesday 11/15*

Do Problem #11 from section 6.4 in your book. (Answer in back, but try it yourself first.)

Solution 2

Matrix A has eigenvalues $\lambda_1 = 2, \lambda_2 = 4$, and eigenvectors $x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. (Notice we've normalized the orthogonal eigenvectors.) So $A = 2x_1x_1^{\mathrm{T}} + 4x_2x_2^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.
Matrix B has eigenvalues $\lambda_1 = 25, \lambda_2 = 0$, and eigenvectors $x_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, x_2 = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ respectively. So $B = 25x_1x_1^{\mathrm{T}} + 0x_2x_2^{\mathrm{T}} = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ (B already has rank 1).

Problem 3 *Wednesday 11/15*

Do Problem #18 from section 6.4 in your book.
Then show the converse: if A has a complete set of orthonormal eigenvectors with real eigenvalues, then it must be symmetric. *(Hint: diagonalize.)*

Solution 3

The nullspace and the *row* space are always perpendicular. But for a symmetric matrix, row space = column space. So if y is an eigenvector for $\lambda \neq 0$ (in the column space), it must be perpendicular to the set of eigenvectors for $\lambda = 0$ (the nullspace). *(And perpendicular to the other eigenspaces $\lambda = \beta$ too — use $A - \beta I$ (also symmetric) instead, and the same argument.)*

Going the other way: If A has a complete set of eigenvectors, we can diagonalize: $A = S\Lambda S^{-1}$. The eigenvectors are orthonormal, so in fact $A = Q\Lambda Q^{-1}$ where $Q^{-1} = Q^{\mathrm{T}}$. And that means $A^{\mathrm{T}} = Q\Lambda^{\mathrm{T}}Q^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$.

Problem 4 *Wednesday 11/15*

Do Problem #27 from section 6.4 in your book.

Solution 4

The other eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (eigenvalue $\lambda = 1 + 10^{-15}$), which makes an angle with the other eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of only $45^o = \pi/4$! *Moral: eigenvectors are very sensitive to roundoff error.*

Problem 5 *Friday 11/17*

Do Problem #7 from section 6.5 in your book.
Use all four tests for each of these: find the pivots, the eigenvalues, the upper-left determinants, and the "quadratic form" $x^{\mathrm{T}}Mx$.

Solution 5

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, M = A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$:

- Elimination gives $\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{9} \end{bmatrix}$, so both pivots are positive. ✓ [1]
- The characteristic polynomial is $\lambda^2 - 14\lambda + 9 = 0$, with roots $\lambda = 7 \pm \sqrt{40}$, both positive. ✓
- The upper-left determinants are $|1| = 1, \det(M) = 9$, both positive. ✓
- The "quadratic form" $x^{\mathrm{T}}(A^{\mathrm{T}}A)x = x_1^2 + 4x_1x_2 + 13x_2^2$ is positive definite (positive for nonzero x): to see this, write it as a sum of two squares, $x^{\mathrm{T}}Ax = 1(x_1 + 2x_2)^2 + 9(x_2)^2$. This can only be zero when $x_2 = 0$ and $x_1 = 0$: otherwise, it must be positive.✓ *(Notice the pivots? That's because we're really writing $M = LDL^T$, so $x^TMx = (Lx)^TD(Lx)$. The coefficients of L give the linear combinations inside the squares, and the pivots on the diagonal of D give the multipliers outside.)*

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, M = A^TA = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$:

- Elimination gives $\begin{bmatrix} \boxed{6} & 5 \\ 0 & \boxed{11/6} \end{bmatrix}$, both pivots positive. ✓
- The eigenvalues are $\lambda = 1, \lambda = 11$, both positive. ✓
- The upper-left determinants are $|6| = 6, \det(M) = 11$, both positive. ✓
- The "quadratic form" $x^{\mathrm{T}}Mx = 6x_1^2 + 10x_1x_2 + 6x_2^2$ is positive definite: write it as $x^{\mathrm{T}}Mx = 6(x_1 + 5x_2)^2 + (11/6)x_2^2$. This can only be zero when $x_2 = 0$ and $x_1 = 0$: otherwise, it must be positive. ✓

[1] Remember: we're doing all four tests here, because the problem asks us to. But any ONE of the tests would be enough to tell us whether $A^{\mathrm{T}}A$ is positive-definite or not — they're all "equivalent".

$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}, M = A^T A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$:

- Elimination gives $\begin{bmatrix} \boxed{2} & 3 & 3 \\ 0 & \boxed{1/2} & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{2} & 3 & 3 \\ 0 & \boxed{1/2} & -1/2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$.⊗
- The characteristic polynomial is $\lambda^3 - 12\lambda^2 + 11\lambda = 0$, which has roots $\lambda = 1, 11, 0$. So one of the eigenvalues is zero. ⊗
- The upper-left determinants are $|2| = 2, \left|\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\right| = 1, \det(M) = 0$. So one of these is zero, too. ⊗
- The "quadratic form" $x^{\mathrm{T}} M x = x_1^2 + 6x_1x_2 + x_2^2 + 6x_1x_3 + 8x_2x_3 + 10x_3^2$ is not positive definite: for instance, take $x = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ to make it zero. If you try writing it as a sum of squares[2], you only get two (nonzero) squares in the sum. *(If $x^{\mathrm{T}} M x = 0$, then $x_1 + (3/2)x_2 + (3/2)x_3 = 0$ and $x_2 - x_3 = 0$, but x_3 doesn't have to be zero!)* ⊗

(This matrix is almost positive-definite: $x^{\mathrm{T}} A x$ isn't always positive, but it's always ≥ 0. We say it's (positive) "semidefinite". Notice that semidefinite matrices are just like positive-definite matrices, except now we allow zeroes in the pivots, eigenvalues, determinants, ...)

Problem 6 *Friday 11/17*

Do Problem #20 from section 6.5 in your book.

Solution 6

(a) Every positive-definite matrix has no zero eigenvalues, so its nullspace is just $\{0\}$.
(b) Any non-identity projection matrix has a zero eigenvalue (or: a nonzero nullspace).
(c) The diagonal entries are the eigenvalues, which are all positive.
(d) The product of negative eigenvalues can be positive. For example, $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ has two negative eigenvalues, but its determinant is $+1$. (It's even diagonal!)

Problem 7 *Friday 11/17*

Do Problem #28 from section 6.5 in your book.
Then sketch the ellipse $x^{\mathrm{T}} A x = 1$ for $\theta = \pi/4$. Draw in the eigenvectors.

Solution 7

(a) $\det(A) = \det(\Lambda) = 10$. (b) The eigenvalues are the diagonal entries of Λ, $\lambda_1 = 2$ and $\lambda_2 = 5$. (c) The eigenvectors are $x_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $x_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, the columns of S. (d) A must be symmetric, because S is orthogonal. A must be positive definite, because all its eigenvalues are positive.

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[2] $M = LDL^{\mathrm{T}}$ where $L = \begin{bmatrix} 1 & & \\ 3/2 & 1 & \\ 3/2 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 1/2 & \\ & & 0 \end{bmatrix}$, so $x^{\mathrm{T}} M x = 2(x_1 + (3/2)x_2 + (3/2)x_3)^2 + (1/2)(x_2 - x_3)^2 [+0x_3^2]$

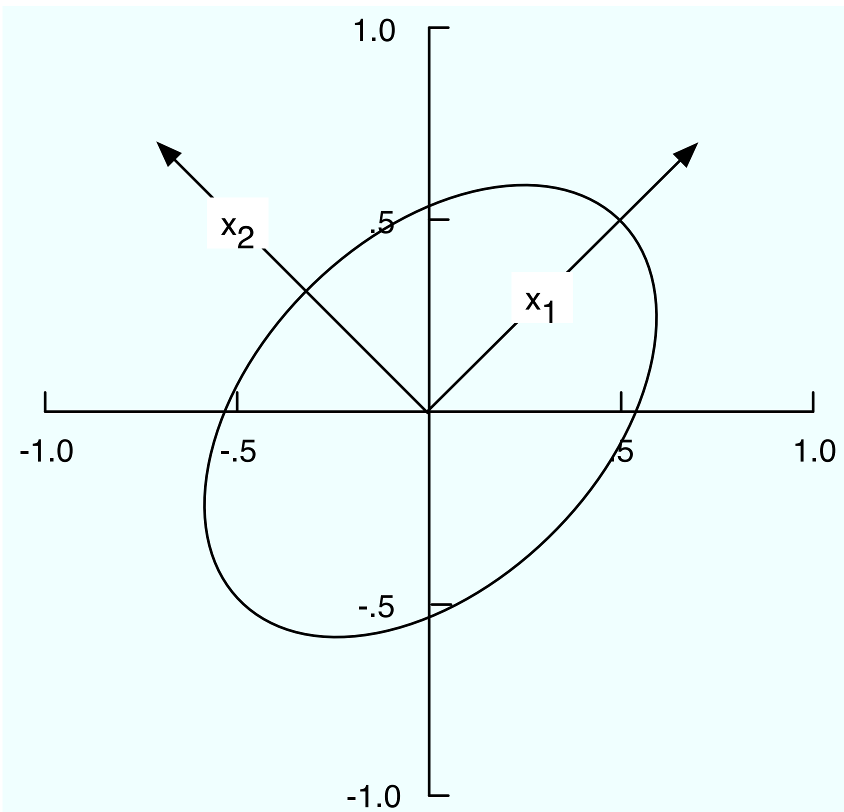

Figure 1: Ellipse for problem #28. Notice that the eigenvectors lie on the principal axes of the ellipse.