18.06 Problem Set 8

Due Wednesday, Nov. 15, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 11/6*

Do Problem #12 from section 8.3 in your book.

Solution 1

The columns of A must sum to 1, so $A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$.

Our theory tells us the steady state is the eigenvector with $\lambda = 1$, and sure enough there is one: $x_1 = (1, 1, 1)$ (or any multiple of x_1) works.

Why is $x_1 = (1, 1, \ldots, 1)$ *a steady state?* The entries of Ax_1 are the sums of each row. *But A is symmetric*, so these are the same as the sums of each column, which are 1. So the entries of Ax_1 are 1, just like the entries of x_1.

Problem 2 *Monday 11/6*

Of 300 million Americans, 60% own their own home and the other 40% rent.

Let's represent these proportions as a vector: $x = \begin{bmatrix} \text{owners} \\ \text{renters} \end{bmatrix} = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$.

Every year, some proportion of renters will buy a house, and some proportion of homeowners will move to a rental. If these proportions remain constant, we can model this with the "Markov process" $x_{k+1} = Ax_k$ for some 2-by-2 Markov matrix A.

Suppose the proportion of homeowners and renters is modeled by this Markov process;

it maintains the steady state $x = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$ above;

and 90 percent of homeowners in any given year k still own a home in year $k + 1$.

Determine A, and estimate how many American renters will buy a home this year.

Solution 2

We know the first column of A (how many homeowners are owners/renters in the following year):

$A = \begin{bmatrix} .90 & ? \\ .10 & ? \end{bmatrix}$

We can calculate the second column of A from the steady state $Ax = x$: $\begin{bmatrix} .90 & a \\ .10 & 1-a \end{bmatrix} \begin{bmatrix} .60 \\ .40 \end{bmatrix} = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$

gives $A = \begin{bmatrix} .90 & .15 \\ .10 & .85 \end{bmatrix}$.

So 15% of American renters, or 6% of Americans, will buy a home, for a total of 18 million new homeowners.

Problem 3 *Wednesday 11/8*

Find the first three nonzero terms in the Fourier series for the period-2π function

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

Then find the lengths of the original function $\|f(t)\|$ and your three-term approximation $\|g(t)\|$, and the distance $\|f(t) - g(t)\|$ between them.

Solution 3

When we expand $f(t)$ as a Fourier series, it looks like $f(t) = a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \ldots$. All we have to do is figure out the coefficients a_i, b_i. This is easy, because *the basis functions* $1, \cos(t), \ldots$ *are orthogonal* — if we take an inner product, all the other terms go away! So, to find a_0, we take the inner product with the basis function 1 —

$$\begin{aligned}
(f,1) = \int_0^{2\pi} a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots\, dt &= \int_0^{2\pi} f(t)\, dt \\
\int_0^{2\pi} a_0\, dt &= \int_0^{\pi} dt \\
2\pi a_0 &= \pi
\end{aligned}$$

so $a_0 = 1/2$.
Similarly for a_1:

$$\begin{aligned}
(f,\cos(t)) = \int_0^{2\pi} (a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots) \cos(t)\, dt &= \int_0^{2\pi} f(t) \cos(t)\, dt \\
\int_0^{2\pi} a_1 \cos(t)\, dt &= \int_0^{\pi} \cos(t) dt \\
\pi a_1 &= 0
\end{aligned}$$

so $a_1 = 0$; in fact, all the cosine coefficients a_k are zero.
Similarly for the sine coefficients b_k:

$$\begin{aligned}
(f,\sin(kt)) = \int_0^{2\pi} (a_0 \cdot 1 + a_1 \cos(t) + b_1 \sin(t) + \ldots) \sin(kt)\, dt &= \int_0^{2\pi} f(t) \sin(kt)\, dt \\
\int_0^{2\pi} a_1 \sin(kt)\, dt &= \int_0^{\pi} \sin(kt) dt \\
\pi b_k &= \left[\frac{-\cos(kt)}{k} \right]_{t=0}^{\pi}
\end{aligned}$$

This gives $b_k = 0$ if k is even, and $b_k = \frac{2}{k\pi}$ if k is odd.
(You could also use the book's formulas to find the coefficients. But this is where they come from.)
So the first three terms of the Fourier series for $f(t)$ are

$$f(t) \approx g(t) = \frac{1}{2} + \frac{2}{\pi} \sin(t) + \frac{2}{3\pi} \sin(3t).$$

Now we find the lengths.
$\|f(t)\|^2 = (f,f) = \int_0^{2\pi} f(t)^2 dt = \int_0^{\pi} dt = \pi$, so $\|f\| = \sqrt{\pi}$.
$\|g(t)\|^2 = (g,g) = (1/2)^2 + (2/\pi)^2 + (2/3\pi)^2$ (the basis vectors are orthogonal!), so $\|g\| = \sqrt{(1/4) + (40/9\pi^2)} = \sqrt{160 + 9\pi^2}/6\pi$.
$\|f(t) - g(t)\|^2 = (f-g, f-g) = \int_0^{2\pi} (f(t) - 1/2 - 2/\pi \sin(t) - 2/3\pi \sin(3t))^2 dt = \ldots$
You could evaluate that integral, but there's an easier way: since g is the orthogonal projection of f into a subspace, the error $f - g$ $(= b_5 \sin(5t) + b_7 \sin(7t) + \ldots)$ is orthogonal to g! So $\|f\|^2 = \|g\|^2 + \|f - g\|^2$ and $\|f - g\|^2 = \sqrt{\pi} - \sqrt{160 + 9\pi^2}/6\pi$.
(It's not obvious this is positive (as lengths should be), but it is.)

Problem 4 *Wednesday 11/8*

Do Problem #1 from section 10.2 in your book.

Solution 4

You can still find lengths by the Pythagorean theorem (since $|a+bi| = \sqrt{a^2+b^2}$):
$\|u\| = \sqrt{(1+1)+(1+1)+(1+4)} = \sqrt{9} = 3$,
and $\|v\| = \sqrt{(0+1)+(0+1)+(0+1)} = \sqrt{3}$.
Or take the dot product (don't forget to conjugate!):
$\|u\| = \sqrt{u^{\mathrm{H}}u} = \sqrt{(1-i)(1=i)+(1+i)(1-i)+(1-2i)(1+2i)} = \sqrt{2+2+5} = 3$,
and $\|v\| = \sqrt{v^{\mathrm{H}}v} = \sqrt{(-i)(+i)+(-i)(+i)+(-i)(+i)} = \sqrt{1+1+1} = \sqrt{3}$.
For complex inner products, *order matters:*
$u^{\mathrm{H}}v = (1-i)i + (1+i)i + (1-2i)i = 2+3i$,
but $v^{\mathrm{H}}u = -i(1+i) - i(1-i) - i(1+2i) = 2-3i$!
(The difference is that $(u^Hv)^H = vu^H$ conjugates u, but v^Hu conjugates v. So the two products are conjugates of each other.)

Problem 5 *Wednesday 11/8*

Do Problem #2 from section 10.2 in your book.

Solution 5

$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$ so $A^{\mathrm{H}} = \begin{bmatrix} 1 & -i \\ -i & 1 \\ -i & -i \end{bmatrix}$.
$A^{\mathrm{H}}A = \begin{bmatrix} 0 & 2 & i+1 \\ 2 & 0 & 1+i \\ 1-i & -i+1 & 2 \end{bmatrix}$
$AA^{\mathrm{H}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$
Both of these are *Hermitian:* their *conjugate* transpose is itself, $M^{\mathrm{H}} = \overline{M}^{\mathrm{T}} = M$.
(The second one is also *real* (and hence *symmetric*): this is just a coincidence.)

Problem 6 *Wednesday 11/8*

Do Problem #17 from section 10.2 in your book.

Solution 6

First find the eigenvalues: $\lambda^2 - 2(\cos\theta)\lambda + 1 = 0$ has roots $\lambda = \cos\theta \pm i\sin\theta$. *Notice that both eigenvalues have* $|\lambda| = 1$, since Q is orthogonal.
Now find the eigenvectors. For $\lambda_+ = \cos\theta + i\sin\theta$, we want a vector x with $\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} x = 0$, such as $x_+ = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Similarly, $\lambda_- = \cos\theta - i\sin\theta$ has eigenvector $x_- = \begin{bmatrix} 1 \\ +i \end{bmatrix}$.
These eigenvectors are automatically orthogonal (that is, $(u_+, u_-) = 1(1) - i(-i) = 1 - 1 = 0$), but we want the columns of U to be ortho*normal*, so we need to divide by the lengths: $u_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $u_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix}$.
Then our factorization is

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}}_{U^{\mathrm{H}}}$$

Problem 7 *Wednesday 11/8*

Do Problem #31 from section 10.2 in your book.
(Hints: U is a ____ matrix, so $U^H U =$ ____ . Λ is a ____ matrix, so $\Lambda^H\Lambda$ and $\Lambda\Lambda^H$ are ____ .)

Solution 7

(Answers to hints: U is unitary, so $U^H U = I$. Λ is diagonal, so $\Lambda^H\Lambda = \Lambda\Lambda^H$.)
$A^{\mathrm{H}}A = (U\Lambda^{\mathrm{H}}U^{\mathrm{H}})(U\Lambda U^{\mathrm{H}}) = U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}}$,
$AA^{\mathrm{H}} = (U\Lambda U^{\mathrm{H}})(U\Lambda^{\mathrm{H}}U^{\mathrm{H}}) = U\Lambda\Lambda^{\mathrm{H}}U^{\mathrm{H}}$,
and since $\Lambda^{\mathrm{H}}\Lambda = \Lambda\Lambda^{\mathrm{H}}$, these are equal.

Problem 8 *Wednesday 11/8*

Do Problem #7 from section 10.3 in your book.

Solution 8

Here's one step of the factorization of the Fourier matrix F_4:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}}_{F_4} = \underbrace{\begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix}}_{\text{shuffle together}} \underbrace{\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}}_{\text{half-size } F\text{'s}} \underbrace{\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{\text{split the inputs}}$$

Now just multiply:

$$\begin{aligned} F_4c &= \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

So $c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (the frequency-space representation of $f(t)$, $1e^{0i\pi t} + 0e^{(1/2)i\pi t} + 1e^{i\pi t} + 0e^{(3/2)i\pi t}$)

becomes $y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ (the time-space representation, $\begin{bmatrix} f(0) = 2 \\ f(1) = 0 \\ f(2) = 2 \\ f(3) = 0 \end{bmatrix}$).

Now do the same for $c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$; we get $c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rightsquigarrow y = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$.

(In other words, $f(t) = 0e^{0i\pi t} + 1e^{(1/2)i\pi t} + 0e^{i\pi t} + 1e^{(3/2)i\pi t}$ has time-space representation $\begin{bmatrix} f(0) = 2 \\ f(1) = 0 \\ f(2) = -2 \\ f(3) = 0 \end{bmatrix}$.)

Problem 9 *Monday 11/13*

Do Problem #16 from section 6.3 in your book.

Solution 9

The power series for e^{kt} is $1 + kt + k^2\frac{t^2}{2} + k^3\frac{t^3}{6} + k^4\frac{t^4}{24} + \ldots$.
Same thing for $e^{At} = 1 + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + A^4\frac{t^4}{24} + \ldots$.
Differentiate: $\frac{d}{dt}e^{At} = A + A^2\frac{\not{2}t}{\not{2}} + A^3\frac{\not{3}t^2}{\not{3}\cdot 2} + A^4\frac{\not{4}t^3}{\not{4}\cdot 6} + \ldots$
which is A times the first four terms above.
(This is almost *a proof that* $\exp(At)$ *is a solution to* $u' = Au$ *— we should check that all the other terms work, too! Fortunately, it's easy to see that the pattern holds.)*

Problem 10 *Monday 11/13*

Do Problem #22 from section 6.3 in your book.
Then solve $u' = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} u$ for initial condition $u(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Is the solution stable as $t \to \infty$? Why or why not?

Solution 10

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}}_{S^{-1}}$$

so

$$\underbrace{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}}_{\exp(\Lambda t)} \underbrace{\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}}_{S^{-1}} = \underbrace{\begin{bmatrix} e^t & -\frac{1}{2}e^t + \frac{1}{2}e^{3t} \\ 0 & e^{3t} \end{bmatrix}}_{\exp(At)}$$

At $t = 0$, $\exp(At)$ reduces to $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, as it should.
Solving $u' = Au$*:*

$$u(t) = \underbrace{\begin{bmatrix} e^t & -\frac{1}{2}e^t + \frac{1}{2}e^{3t} \\ 0 & e^{3t} \end{bmatrix}}_{\exp(At)} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{u(0)} = \begin{bmatrix} e^t + e^{3t} \\ 2e^{3t} \end{bmatrix}$$

As $t \to \infty$ both components go to infinity, so this is *not stable*.