18.06 Problem Set 7

Due Wednesday, Nov. 8, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 10/25*

I've started writing Matlab code to compute the cofactor matrix C of a random 4-by-4 matrix A. Finish it for me, then run it in Matlab, and then show it works by comparing C to `det(A)*inv(A)`.

```
A = rand(4);        % Pick a random 4-by-4 matrix A
C = zeros(?);       % C is a matrix of size ..., we'll fill in the entries later
for i=1:4           % For each of the rows
  for j=???         % and each of the columns:
    B = A;          % Make a copy of A, and for this copy
    B(i,:)=[];      % remove row i
    ???;            % and column j
    C(i,j)=???;     % then cofactor entry (i,j) is ...  of B.
  end               %
end                 %
C                   % print C
```

Problem 2 *Wednesday 10/25*

Do Problem #13 from section 5.3 in your book.

Problem 3 *Wednesday 10/25*

I give you a pyramid with a triangular base (i.e., a tetrahedron). The vertices of the base are on the plane $z = 0$, at $(x, y) = (1, 1), (1, -1), (-1, 1)$. The top vertex is at $(x, y, z) = (0, 0, 2)$. Find the surface area (excluding the base) and the volume. *(Hint: Just as the area of a triangle is* $1/2! = 1/2$ *the volume of the corresponding parallelogram, the volume of a tetrahedron is* $1/3! = 1/6$ *the volume of the corresponding box.)*

Problem 4 *Friday 10/27*

Consider the matrix $M = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -14 & -6 & -9 & -7 \\ -2 & -1 & -2 & -1 \\ 8 & 1 & 7 & 4 \end{bmatrix}$.

(a) If one eigenvector is $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, find its eigenvalue λ_1.

(b) $\det(M) = 0$. Tell me another eigenvalue λ_2, and how you know.

(c) Given the eigenvalue $\lambda_3 = -1$, write down a linear system $Ax = b$ which can be solved to find x_3.

(d) What is the trace of A? What is λ_4? How do you know?

Problem 5 *Friday 10/27*

Give a 2-by-2 matrix for each. *(Hint: diagonalizing* $A = S\Lambda S^{-1}$ *may help.)*

(a) an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 1$, and an eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue $\lambda = 5$

(b) one eigenvalue is $1+i$, one eigenvalue is $1-i$, all entries of A are real numbers. What are the eigenvectors of your matrix?
(c) both eigenvalues are 3, and A is diagonalizable. What are the eigenvectors?
(d) both eigenvalues are 3, and A is not diagonalizable. What are the eigenvectors?

Problem 6 *Friday 10/27*

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Then write $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix.

Problem 7 *Friday 11/3*

Do Problem #20 from section 6.2 in your book.

Problem 8 *Friday 11/3*

The power method. We know one way to find eigenvectors — look for the roots of $\det(A-\lambda I)=0$, and then solve $(A-\lambda I)x=0$. For large matrices, *this is hard* — determinants are hard, and factoring polynomials is hard. Here's another way.
(a) Suppose $A = S\Lambda S^{-1}$, where S's columns are the eigenvectors x_i of A. Then $A^2 =$ ____ . $A^{100} =$ ____ .
(b) If v is any vector, we can write it as a linear combination of the eigenvectors: $v = Sc = c_1x_1 + \ldots + c_nx_n$. If x_1 has eigenvalue λ_1, etc., then $Av = ASc =$ ____ , $A^2v =$ ____ , and $A^{100}v =$ ____ .
(c) If λ_1 is the largest eigenvalue, which term in A^k is growing the fastest? If λ_1 is twice as large as any of the other λ_i, I would expect that term in A^{100} to be about ____ times as large as any of the others. So A^{100} is very close to ____ . What if λ_1 is only 5% larger than the others?
(d) Now go to Matlab, and start with a random 10-by-10 matrix A in Matlab (`A=rand(10)` works). Pick a random 10-element vector v (`v=rand(10,1)` or pick your own!), and calculate `u=(A^100)*v`.
(e) Let's see if u really is an eigenvector. One way you could do this is to divide each element of Au by the corresponding element of u, like this: `(A*u)./u` — here `x./y` gives the vector whose jth entry is x_j/y_j. Is u an eigenvector? How can you tell?
(You can actually use this "power method" to find any eigenvalue, not just the largest. For example, to find the smallest eigenvalue of A, look for the largest eigenvalue of A^{-1}. Or find the eigenvalue closest to c by looking for the largest eigenvalue of $(A-cI)^{-1}$ — by varying c, you can find all eigenvalues of A.)

Problem 9 *Friday 11/3*

Every projection matrix satisfies $P^2 = P$. *(Pb is in the subspace, so $P(Pb) = Pb$.)*
Do Problem #29 from section 6.2 in your book.
What are the eigenvalues of a projection matrix?