# 18.06 Problem Set 7

Due Wednesday, Nov. 8, 2006 at **4:00 p.m.** in 2-106

**Problem 1** *Wednesday 10/25*

I've started writing Matlab code to compute the cofactor matrix $C$ of a random 4-by-4 matrix $A$. Finish it for me, then run it in Matlab, and then show it works by comparing $C$ to `det(A)*inv(A)`.

```
A = rand(4);      % Pick a random 4-by-4 matrix A
C = zeros(?);     % C is a matrix of size ..., we'll fill in the entries later
for i=1:4         % For each of the rows
  for j=???       % and each of the columns:
    B = A;        % Make a copy of A, and for this copy
    B(i,:)=[];    % remove row i
    ???;          % and column j
    C(i,j)=???;   % then cofactor entry (i,j) is ...  of B.
  end             %
end               %
C                 % print C
```

***Solution 1***

```
>> A=rand(4);
>> C=zeros(4); % Any 4-by-4 matrix will do here.
>> for i=1:4
for j=1:4
B=A;
B(i,:)=[];
B(:,j)=[];
C(i,j)=(-1)^(i+j)*det(B);
end
end
>> C
C =
    0.2614   -0.0880   -0.2358    0.1510
   -0.2714    0.1400    0.1644   -0.0021
   -0.0542    0.1969    0.1795   -0.2944
   -0.0766   -0.1403    0.1586    0.0733

>> det(A)*inv(A)
ans =
    0.2614   -0.2714   -0.0542   -0.0766
   -0.0880    0.1400    0.1969   -0.1403
   -0.2358    0.1644    0.1795    0.1586
    0.1510   -0.0021   -0.2944    0.0733

>> C'-ans
ans =
   1.0e-16 *

    0.5551   -0.5551    0.2082   -0.2776
         0    0.5551         0    0.2776
```

```
     0   -0.5551          0          0
     0   -0.0651     0.5551          0
```

`C'` isn't exactly equal to `det(A)*inv(A)` because of roundoff, but it's close (within $1.0 \times 10^{-16}$, in this case).

**Problem 2** *Wednesday 10/25*

Do Problem #13 from section 5.3 in your book.

***Solution 2***

We know $\det(A)\det(A^{-1}) = 1$, and we know that both determinants are integers (since all the entries of $A$ and $A^{-1}$ are integers). If one of them were greater than 1 (in absolute value), then the other would have to be less than 1, which isn't possible.

**Problem 3** *Wednesday 10/25*

I give you a pyramid with a triangular base (i.e., a tetrahedron). The vertices of the base are on the plane $z = 0$, at $(x, y) = (1, 1), (1, -1), (-1, 1)$. The top vertex is at $(x, y, z) = (0, 0, 2)$. Find the surface area (excluding the base) and the volume. *(Hint: Just as the area of a triangle is* $1/2! = 1/2$ *the volume of the corresponding parallelogram, the volume of a tetrahedron is* $1/3! = 1/6$ *the volume of the corresponding box.)*

***Solution 3***

*Surface area:* I used cross products.
If $u_1 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^T, u_2 = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}^T, u_3 = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}^T$ are the three edges from the top vertex to the base vertices, then the three faces have areas
$\frac{1}{2}\|u_1 \times u_2\| = \frac{1}{2}\| -4i - 2j\| = \sqrt{5}$
$\frac{1}{2}\|u_1 \times u_3\| = \sqrt{5}$, by symmetry
$\frac{1}{2}\|u_2 \times u_3\| = \frac{1}{2}\|4i + 4j + 2k\| = 3$
So the total surface area is $3 + 2\sqrt{5}$.
*Volume:* The (signed) volume is
$\frac{1}{6}\det\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = 8/6$
so the volume of the tetrahedron is $4/3$.

**Problem 4** *Friday 10/27*

Consider the matrix $M = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -14 & -6 & -9 & -7 \\ -2 & -1 & -2 & -1 \\ 8 & 1 & 7 & 4 \end{bmatrix}$.

(a) If one eigenvector is $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, find its eigenvalue $\lambda_1$.
(b) $\det(M) = 0$. Tell me another eigenvalue $\lambda_2$, and how you know.
(c) Given the eigenvalue $\lambda_3 = -1$, write down a linear system $Ax = b$ which can be solved to find $x_3$.
(d) What is the trace of $A$? What is $\lambda_4$? How do you know?

***Solution 4***

(a) $Mx_1 = x_1$ so $\lambda_1 = 1$.
(b) The determinant is the product of the eigenvalues, so one of the eigenvalues must be $\lambda_2 = 0$. (Alternatively: $M$ is singular, so (since it's square) it must have a nontrivial nullspace.)
(c) $x_3$ satisfies $(M - \lambda_3 I)x_3 = 0$, or in more detail, $\begin{bmatrix} 3 & 2 & 1 & 1 \\ -14 & -5 & -9 & -7 \\ -2 & -1 & -1 & -1 \\ 8 & 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} \\ x_3 \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
(d) The trace of $M$ is $(2) + (-6) + (-2) + (4) = -2$. *(So the trace of $A = M - 3I$ is $(-1) + (-9) + (-5) + 1 = -14$. I meant to ask for the trace of $M$, not $A$ — sorry!)* This is the sum of the four eigenvalues, and we know the other three are 0, 1, and $-1$, so the fourth eigenvalue must be $\lambda_4 = -2$. *(Or you could find the roots of* $\det(A - \lambda I)$*...)*

**Problem 5** *Friday 10/27*

Give a 2-by-2 matrix for each. *(Hint: diagonalizing $A = S\Lambda S^{-1}$ may help.)*
(a) an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 1$, and an eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue $\lambda = 5$
(b) one eigenvalue is $1 + i$, one eigenvalue is $1 - i$, all entries of $A$ are real numbers. What are the eigenvectors of your matrix?
(c) both eigenvalues are 3, and $A$ is diagonalizable. What are the eigenvectors?
(d) both eigenvalues are 3, and $A$ is not diagonalizable. What are the eigenvectors?

***Solution 5***

(a) We know the eigenvectors $S$ and eigenvalues $\Lambda$, so
$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$
(You could also put the eigenvectors in the other order, but in this case you'll get the same matrix.)
(b) We need $A$ to have determinant 2 and trace 2. How about $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$?
Now find the eigenvectors. For $(1 + i)$, $\begin{bmatrix} 1 - (1+i) & -1 \\ 1 & 1 - (1+i) \end{bmatrix}$ has nullspace $\left\{ c \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}$. For $(1 - i)$, do the same thing *(or use symmetry — take the complex conjugate)* to get the set of eigenvectors $\left\{ c \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}$.
(c) $\Lambda = 3I$, so $S\Lambda S^{-1} = 3SS^{-1}$ must be $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Every vector in $\mathbb{R}^2$ is an eigenvector!
(d) $\begin{bmatrix} 3 & c \\ 0 & 3 \end{bmatrix}$ will work, for any $c$. It only has a one-dimensional "eigenspace": all multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (There are other possibilities, too.)

**Problem 6** *Friday 10/27*

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Then write $A = S\Lambda S^{-1}$, where $\Lambda$ is a diagonal matrix.

***Solution 6***

The characteristic polynomial $\det(A - \lambda I) = \det \begin{bmatrix} 0 - \lambda & 1 & 0 & 1 \\ 0 & 1 - \lambda & -2 & -7 \\ 0 & 0 & -1 - \lambda & -1 \\ 0 & 0 & 0 & 2 - \lambda \end{bmatrix}$ is $(-\lambda)(1 - \lambda)(-1 - \lambda)(2 - \lambda)$, so its roots (the eigenvalues) are $0, -1, 1, -2$.

Now solve $(A-\lambda I)x=0$ for each eigenvalue $\lambda$:

$\lambda=0$: Any multiple of $x_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is in the nullspace $\{x : Ax=0\}$.

$\lambda=1$: Any multiple of $x_2=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ is in the nullspace of $(A-I)$.

$\lambda=-1$: Here $(A+I)=\begin{bmatrix}1&1&0&1\\0&2&-2&-7\\0&0&0&1\\0&0&0&-1\end{bmatrix}$. The first three columns are linearly dependent: $x_3=\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}$ (or any multiple of it) is a solution to $(A+I)x=0$, by inspection.

$\lambda=-2$: We solve the system $(A+2I)x=0$, or $\begin{bmatrix}2&1&0&1\\0&3&-2&-7\\0&0&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$. Any (nonzero) solution will do, so pick $d=1$; then by back-substitution $c=1$, $b=3$, and $a=-2$ so $x_4=\begin{bmatrix}-2\\3\\1\\1\end{bmatrix}$ generates the $\lambda=-2$ eigenspace.

Now we can diagonalize: $$\underbrace{\begin{bmatrix}0&1&0&1\\0&1&-2&-7\\0&0&-1&-1\\0&0&0&2\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&1&-1&-2\\0&1&1&3\\0&0&1&1\\0&0&0&1\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}0&&&\\&1&&\\&&-1&\\&&&-2\end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix}1&1&-1&-2\\0&1&1&3\\0&0&1&1\\0&0&0&1\end{bmatrix}^{-1}}_{S^{-1}}$$

**Problem 7** *Friday 11/3*

Do Problem #20 from section 6.2 in your book.

***Solution 7***

I found the two eigenvalues $\lambda_1=1,\lambda_2=\frac{1}{5}$ corresponding to the eigenvectors $x_1=\begin{bmatrix}1\\1\end{bmatrix}, x_2=\begin{bmatrix}1\\-1\end{bmatrix}$.

So I got $A=\underbrace{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}1&0\\0&1/5\end{bmatrix}}_{\Lambda}\underbrace{\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}}_{S^{-1}}$.

$\Lambda^k=\begin{bmatrix}1&0\\0&(1/5)^k\end{bmatrix}$ goes to $\Lambda^\infty=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ as $k\to\infty$, so $A^k=S\Lambda^kS^{-1}$ goes to $A^\infty=S\Lambda^\infty S^{-1}=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$. The columns are just $x_1$!

- *If you chose different eigenvectors* $\begin{bmatrix}c\\c\end{bmatrix},\begin{bmatrix}d\\-d\end{bmatrix}$ *for* $\lambda_1$ *and* $\lambda_2$*, you'd get* $S=\begin{bmatrix}c&d\\c&-d\end{bmatrix}$ *but* $\Lambda$ *would be the same. Or if you put them in the reverse order* $\lambda_1=\frac{1}{5},\lambda_2=1$ *you'd get* $S=\begin{bmatrix}d&c\\-d&c\end{bmatrix}$ *and* $\Lambda=\begin{bmatrix}1/5&0\\0&1\end{bmatrix}$*. Unlike our other key factorizations* $A=LU$ *and* $A=QR$*, diagonalization* $A=S\Lambda S^{-1}$ *isn't unique.*
- *Did you notice that* $A$ *is a Markov matrix? That means we didn't have to find all the eigenvectors to know what happens to* $A^\infty$ *— we could use our Markov-matrix theory instead.*
- *Did you notice that our eigenvectors* $x_1$ *and* $x_2$ *were orthogonal? In fact, this always happens for symmetric matrices — we can even pick the eigenvectors to be orthonormal, so* $A=Q\Lambda Q^{-1}$*!*

*There's more to say about eigenvectors of symmetric matrices, as we'll see next Wednesday.*

**Problem 8** *Friday 11/3*

*The power method.* We know one way to find eigenvectors — look for the roots of $\det(A-\lambda I)=0$, and then solve $(A-\lambda I)x=0$. For large matrices, *this is hard* — determinants are hard, and factoring polynomials is hard. Here's another way.
(a) Suppose $A = S\Lambda S^{-1}$, where $S$'s columns are the eigenvectors $x_i$ of $A$. Then $A^2 =$ ____ . $A^{100} =$ ____ .
(b) If $v$ is any vector, we can write it as a linear combination of the eigenvectors: $v = Sc = c_1x_1 + \ldots + c_nx_n$. If $x_1$ has eigenvalue $\lambda_1$, etc., then $Av = ASc =$ ____ , $A^2v =$ ____ , and $A^{100}v =$ ____ .
(c) If $\lambda_1$ is the largest eigenvalue, which term in $A^kv$ is growing the fastest? If $\lambda_1$ is twice as large as any of the other $\lambda_i$, I would expect that term in $A^{100}v$ to be about ____ times as large as any of the others. So $A^{100}v$ is very close to ____ . What if $\lambda_1$ is only 5% larger than the others?
(d) Now go to Matlab, and start with a random 10-by-10 matrix $A$ in Matlab (`A=rand(10)` works). Pick a random 10-element vector $v$ (`v=rand(10,1)` or pick your own!), and calculate `u=(A^100)*v`.
(e) Let's see if $u$ really is an eigenvector. One way you could do this is to divide each element of $Au$ by the corresponding element of $u$, like this: `(A*u)./u` — here `x./y` gives the vector whose $j$th entry is $x_j/y_j$. Is $u$ an eigenvector? How can you tell?
*(You can actually use this "power method" to find any eigenvalue, not just the largest. For example, to find the smallest eigenvalue of $A$, look for the largest eigenvalue of $A^{-1}$. Or find the eigenvalue closest to $c$ by looking for the largest eigenvalue of $(A-cI)^{-1}$ — by varying $c$, you can find all eigenvalues of $A$.)*

***Solution 8***

(a) If $A = S\Lambda S^{-1}$, then $A^2 = S\Lambda^2 S^{-1}$ and $A^{100} = S\Lambda^{100}S^{-1}$.
(b) Then if $v = Sc$, then $Av = ASc = S\Lambda c = \lambda_1c_1x_1+\lambda_2c_2x_2+\ldots+\lambda_nc_nx_n$, $A^2v = A^2Sc = S\Lambda^2c = \lambda_1^2c_1x_1+\lambda_2^2c_2x_2+\ldots+\lambda_n^2c_nx_n$, and $A^{100}v = A^{100}Sc = S\Lambda^{100}c = \lambda_1^{100}c_1x_1+\lambda_2^{100}c_2x_2+\ldots+\lambda_n^{100}c_nx_n$.
(c) If $\lambda_1$ is the largest eigenvalue, the term $\lambda_1^kc_1x_1$ is growing the fastest. If $\lambda_1 \geq 2\lambda_j$ then $\lambda_1^{100} \geq 2^{100}\lambda_j^{100}$, so the $\lambda_1$-term will be about $2^{100}$ times larger than any of the other terms. *(Assuming the $c_i$ are similar in size.)* Even if $\lambda_1 \geq 1.05\lambda_j$, then $\lambda_1^{100} \geq (1.05)^{100}\lambda_j^{100}$ so the $\lambda_1$-term is about $(1.05)^{100} \approx 131$ times as large as the next-largest term.
(d)

```
>> A=rand(10)
A =
    0.6154    0.0579    0.0153    0.8381    0.1934    0.4966    0.7271    0.7948    0.1365    0.5828
    0.7919    0.3529    0.7468    0.0196    0.6822    0.8998    0.3093    0.9568    0.0118    0.4235
    0.9218    0.8132    0.4451    0.6813    0.3028    0.8216    0.8385    0.5226    0.8939    0.5155
    0.7382    0.0099    0.9318    0.3795    0.5417    0.6449    0.5681    0.8801    0.1991    0.3340
    0.1763    0.1389    0.4660    0.8318    0.1509    0.8180    0.3704    0.1730    0.2987    0.4329
    0.4057    0.2028    0.4186    0.5028    0.6979    0.6602    0.7027    0.9797    0.6614    0.2259
    0.9355    0.1987    0.8462    0.7095    0.3784    0.3420    0.5466    0.2714    0.2844    0.5798
    0.9169    0.6038    0.5252    0.4289    0.8600    0.2897    0.4449    0.2523    0.4692    0.7604
    0.4103    0.2722    0.2026    0.3046    0.8537    0.3412    0.6946    0.8757    0.0648    0.5298
    0.8936    0.1988    0.6721    0.1897    0.5936    0.5341    0.6213    0.7373    0.9883    0.6405

>> v=rand(10,1)
v =
    0.2091
```

```
    0.3798
    0.7833
    0.6808
    0.4611
    0.5678
    0.7942
    0.0592
    0.6029
    0.0503

>> u=A^100*v
u =
   1.0e+71 *

    2.1393
    2.5237
    3.2093
    2.5616
    1.9181
    2.5731
    2.4894
    2.6112
    2.1445
    2.8755
```

*(If you didn't want to print that big matrix, you could use* `>> A=rand(10);` *(with a semicolon) — the semicolon tells Matlab not to print a response.)*

*Any multiple of $u$ is also an eigenvector, so we could throw away that extra factor* $1.0 \times 10^{71}$.

(e) Here's what I got: so $u$ is an eigenvector (with eigenvalue $\lambda_1 = 5.2173$), because $Au = 5.2173u$.

```
>> (A*u)./u
ans =
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
    5.2173
```

**Problem 9** *Friday 11/3*

Every projection matrix satisfies $P^2 = P$. *(Pb is in the subspace, so $P(Pb) = Pb$.)*
Do Problem #29 from section 6.2 in your book.
What are the eigenvalues of a projection matrix?

***Solution 9***

If $y = Ax$ is any vector in the column space of $A$, then $Ay = A^2x = Ax = y$, so every vector in the column space is an eigenvector for $\lambda = 1$. *(r independent vectors form a basis for this "eigenspace")*

The nullspace, as always, is the set of eigenvectors for $\lambda = 0$. *(n-r independent vectors here)*

A projection matrix always has $P^2 = P$, so its eigenvalues are 1 and 0. *(with multiplicities $r$ and $n - r$, respectively)*