18.06 Problem Set 6

Due Wednesday, Oct. 25, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 10/18*

Some theory of orthogonal matrices:
(a) Show that, if two matrices Q_1 and Q_2 are orthogonal, then their product Q_1Q_2 is orthogonal.[1]
(b) Show that, if Q is a *square* orthogonal matrix, then its transpose Q^{T} is also orthogonal. *(Hint: Q has an inverse. What is Q^{-1}?)*
(c) Is the transpose of a *non-square* orthogonal matrix still orthogonal? Explain why or why not.

Problem 2 *Wednesday 10/18*

(a) Do Gram-Schmidt elimination on A=$\begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & -1 \\ 3 & -5 & 9 \end{bmatrix}$ to find $A = QR$.

(b) *(You can do this by hand, but I recommend Matlab.)* Find A^{T}A, and then factor this (symmetric) matrix in your choice of two ways:

- LDU-factorization $A^{\mathsf{T}}A = LDL^{\mathsf{T}}$ ($U = L^{\mathsf{T}}$, since $A^{\mathsf{T}}A$ is symmetric)[2]
- Cholesky factorization $A^{\mathsf{T}}A = LL^{\mathsf{T}}$ (a variant of LDL^{T}; the L is different!)[3]

(c) How are L^{T} and R related? *Gram-Schmidt on A is just elimination on $A^{\mathsf{T}}A$!*

Problem 3 *Wednesday 10/18*

(a) Write down the matrix P representing the projection onto the plane *perpendicular* to $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.
(Hint: $P = I - P_1$, where P_1 is the projection ____ .)
(b) Now write down the matrix Q representing the *reflection* through that plane. *(Q is sometimes called a "Householder matrix".)* $Q = I - 2vv^{\mathsf{T}}$ for some vector $v =$ ____ .
(c) Show Q is an orthogonal matrix.

Problem 4 *Friday 10/20*

Do Problem #32 from section 5.1 in your book. *(Uses Matlab.)*

Problem 5 *Friday 10/20*

Do Problem #24 from section 5.1 in your book.

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[1] Remember that an "orthogonal matrix" is really an *orthonormal* matrix; its columns are orthogonal *and* normalized.

[2] The `slu.m` Teaching Code only gives you $A^{\mathsf{T}}A = LU$; you'll have to calculate D on your own. Here's one way: extract the diagonal of U into a vector d with `d = diag(U)`, then make a diagonal matrix out of d with `D=diag(d)` *(same function name, different functions!)*.

[3] If D has only positive pivots, then we can take its square root and write LDL^{T} even more simply, as $(L\sqrt{D})(\sqrt{D}^{\mathsf{T}}L^{\mathsf{T}}) = L_1L_1^{\mathsf{T}}$, where $L_1 = (L\sqrt{D})$. That's the *Cholesky factorization*, which you can get in Matlab by `L=chol(A'A)`.

Problem 6 *Monday 10/23, but you can start on Friday*

Do Problem #14 from section 5.1 in your book.
Now compute these determinants using the big formula (with $n!$ terms) or cofactor expansion (your choice). Which is easier?
(The determinants are $\det(A) = 36, \det(B) = 5$, *if you want to check your work. Note that* $\det(A)$ *is wrong in the back of the book—sorry!)*

Problem 7 *Monday 10/23, but you can start on Friday*

Suppose we fit the quadratic $y = C + Dt + Et^2$ to three points $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ by least-squares.
(a) Write down the least-squares matrix V. $V \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ *(V is called the "Vandermonde matrix".)*
(b) Find $\det V$ by row operations.
(c) Now write down the big formula (with 3! terms) for $\det V$.
(d) Here's a trick for finding $\det V$ easily: we see from the big formula that $\det V$ is a polynomial in a_1, a_2, a_3, and all 3! terms have degree ____. Now find the factors of $\det V$. The first two rows are equal when ____, so when ____, $\det V = 0$. Name a factor of $\det V$: ____. Now name two more factors of $\det V$, for the other two pairs of rows: ____, ____. How do you know any remaining factor of $\det(V)$ is constant? Now find the constant, and you're done!
(e) When can we fit a quadratic *exactly* through three points?

Problem 8 *Monday 10/23*

Do Problem #25 from section 5.2 in your book.

Problem 9 *Monday 10/23*

Do Problem #14 from section 5.2 in your book.