18.06 Problem Set 3

Due Wednesday, Sept. 27, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 9/18*

Look at Worked Example 3.1B from section 3.1 in your book.
For each of the vector spaces $\mathbf{V_1}$ through $\mathbf{V_4}$,
describe a subspace, different from the examples in the text, in two different ways:
all combinations of ... = all solutions to ...

Solution 1

Your examples will vary, but here are some possibilities:

1. All combinations of $(1,1,0,0)$ and $(1,1,1,1)$ =
all solutions (v_1,v_2,v_3,v_4) to $v_1 = v_2$ and $v_3 = v_4$
2. All combinations of $(1,0,0,-1)$ (that is, all multiples $(c,0,0,-c)$) =
all solutions (v_1,v_2,v_3,v_4) to $v_1 + v_4 = 0$, $v_2 = 0$, and $v_3 = 0$
3. All combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ =
all solutions $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $b = c$ and $d = 2a$
4. All combinations of x, x^2, and x^3 (that is, all cubics with $y(0) = 0$) =
all solutions $y(x)$ to $d^4y/dx^4 = 0$ and $y(0) = 0$.

Problem 2 *Monday 9/18*

(a) Do Problem #19 from section 3.1 in your book.
(b) Also describe the nullspaces of each matrix.

Solution 2

(a) $C(A)$ = the x-axis (given by the equations $y = 0, z = 0$), in the 3-D space $\mathbb{R}^3$.
$C(B)$ = the $xy-$plane ($z = 0$).
$C(C)$ = the line $2x = y, z = 0$.
(b) $N(A)$ = the line $2x + y = 0$, in the plane $\mathbb{R}^2$.
$N(B) = Z$, the zero vector (a point in $\mathbb{R}^2$).
$N(C)$ = the y-axis $x = 0$.

Problem 3 *Wednesday 9/20*

Suppose the m-by-n matrix A ($m < n$) has a *right inverse* B, that is, a matrix B such that $AB = I$, the identity.
(a) What must the dimensions of B and of I be?
(b) Try calculating B in Matlab: let $A = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix}$ and find `A\I`. (The identity is `eye(k)` in Matlab.)
(c) Now try calculating B another way, with `rref([A I])`. (This is the reduced-row echelon form, the result of Gauss-Jordan elimination.) What do you get? Now state another, different, B with $AB = I$. (*Hint:* Not all the rows of B are shown, unlike the square case.)
(d) Why can't there be a left inverse $CA = I$? And what would the dimensions of C and I be if there were?

Solution 3

(a) B is n-by-m, and I is m-by-m.

(b) I get $\begin{bmatrix} 1/2 & 5/4 \\ 0 & 0 \\ 0 & 1/2 \end{bmatrix}$ for B.

(c) I get $\begin{bmatrix} 1 & 0 & 1/2 & 1/2 & 3/2 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix}$ for `rref([A I])`. The last two columns suggest to me that $B = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$, and indeed $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

OK, that's cheating: I just *guessed* that the last row of B is zero. *(Sometimes guessing is the best way to find an answer!)* Here's a different route:

1. Take $AB = I$ column by column: the first column of I gives us $Ax_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the second column of I gives us $Ax_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *(I'm writing B by columns, $B = [x_1 x_2]$).*

2. *This is just $Ax = b$,* for the first column and $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then for the second column and $b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We know how to solve this: do elimination! Gauss-Jordan elimination on $\begin{bmatrix} A & b_1 & b_2 \end{bmatrix}$ is just `rref([A I])`, and we can read off a particular solution for each column.

3. For the first column, we have $\begin{bmatrix} \boxed{1} & 0 & 1/2 & 1/2 \\ 0 & \boxed{1} & -2 & 0 \end{bmatrix}$ —
our best particular solution (set "free" variable to zero) is $x_1 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$.

4. For the second column, we have $\begin{bmatrix} \boxed{1} & 0 & 1/2 & 3/2 \\ 0 & \boxed{1} & -2 & -1 \end{bmatrix}$ —
our best particular solution (set "free" variable to zero) is $x_2 = \begin{bmatrix} 3/2 \\ -1 \\ 0 \end{bmatrix}$.

5. Put our two elimination solutions together to get $B = [x_1 x_2] = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$.

(d) If $CA = I$, then C has to be n-by-m, and I is the n-by-n identity.
Here are two reasons we can't have a left inverse (either will suffice):

1. Multiply both sides by B on the right: then we have $CAB = B$. But since we already have a right inverse $AB = I$, so this gives $C = B$. We know that can't happen, because C and B are different sizes.

2. A has rank m (or less), C has rank m (or less), so CA has rank at most m also. So it can't equal I, which has full rank n.

Moral: only square matrices can have both left and right inverses. (And if both inverses exist, they're the same!)

Problem 4 *Wednesday 9/20*

Do Problem #23 from section 3.2 in your book.

Solution 4

$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ is one matrix that has $N(A) = \left\{ c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} : c \in R \right\}$ and $C(A) = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} : a, b \in R \right\}$.

Problem 5 *Wednesday 9/20*

Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Describe the nullspace of the matrix vv^T geometrically.

Solution 5

$N(vv^\mathsf{T}) = \{u : v^\mathsf{T} u = 0\}$, the plane $x + 2y + 3z = 0$ perpendicular to v.

Problem 6 *Friday 9/22*

Do Problem #3 from section 3.3 in your book.

Solution 6

The row reduced forms are:

$\texttt{rref(A)}= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\texttt{rref(}[A \quad A]\texttt{)}= \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\texttt{rref(}\begin{bmatrix} A & A \\ A & 0 \end{bmatrix}\texttt{)}= \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Problem 7 *Friday 9/22*

Let $A= \begin{bmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{bmatrix}$.[1]

(a) Reduce A to (ordinary) echelon form.
(b) What are the pivots? What are the free variables?
(c) Now reduce A to row-reduced echelon form.
(d) Give the special solutions. What is the nullspace $N(A)$?
(e) What is the rank of A?
(f) Give the complete solution to $Ax = b$, where $b = A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution 7

(a) $U = \begin{bmatrix} \boxed{-1} & 2 & 5 & 0 & -5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-1} & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) The pivots are -1, 5, and -1 (as marked, in columns 1, 2, and 4). The other columns are free, corresponding to the third and fifth variables (say, x_3 and x_5).

(c) $R = \begin{bmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

[1] Modified 12/24—thanks to Laura Garrity for pointing out the original solution's error. Hopefully it's fixed now.

(d) Set $x = \begin{bmatrix} * \\ * \\ 1 \\ * \\ 0 \end{bmatrix}$, then solve $Ax = 0$ (or $Rx = 0$) to get $x = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for our first special solution.

Set $x = \begin{bmatrix} * \\ * \\ 0 \\ * \\ 1 \end{bmatrix}$, then solve $Ax = 0$ (or $Rx = 0$) to get $x = \begin{bmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{bmatrix}$ for our second special solution.

The nullspace is the set of their linear combinations, $\left\{ x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{bmatrix} \right\}$.

(e) The rank of A is $r = 3$.

(f) One x that obviously works in $Ax = A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ itself, for our particular solution.

Then the complete solution is $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 1 \\ 0 \\ -6 \\ 1 \end{bmatrix}$.

Problem 8 *Friday 9/22*

Do Problem #5 from section 3.4 in your book.

Solution 8

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{bmatrix}$$

so we need $b_3 - b_2 - 2b_1 = 0$ in order to have a solution to $Ax = b$.
Special solutions: Setting the free variable $x_3 = 1$, we get $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as the solution to $Ax = 0$.
Particular solution: Setting the free variable $x_3 = 0$, we get $\begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix}$ as a particular solution to $Ax = b$.
Complete solution: Putting them together, we get the complete solution $x = \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ for any x_3. (Provided $b_3 - b_2 - 2b_1 = 0$.)

Problem 9 *Friday 9/22*

Do Problem #13 from section 3.4 in your book.
(Answer in back of book, but try to do it yourself first.)

Solution 9

(a) The particular solution x_p is always multiplied by 1
(b) Any solution can be the particular solution
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
(d) The "homogeneous" solution in the nullspace is $x_n = 0$ when A is invertible.

Problem 10 *Friday 9/22*

Do Problem #32 from section 3.4 in your book.

Solution 10

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{-1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and the augmented matrix $[U \quad c] = \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 \\ 0 & 0 & 0 & b_4 - 2b_2 + b_1 \end{bmatrix}$.

There's only one free variable in U, namely x_3

Special solution: take $x_3 = 1$, find that soln. to $Ux = 0$. Back-substitution gives $x = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$.

Since L is invertible, this is also the solution to $Ax = (LUx = L0 =)0$.

Particular solution: choose $x_3 = 0$, find that soln. to $Ux = c$ (hence $Ax = b$).

For $b = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ gives $x = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$. For $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ there is no solution, since $b_4 - 2b_2 + b_1 \neq 0$.