18.06 Problem Set 2

Due Wednesday, Sept. 20, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 9/11*

Do Problem #7 from section 2.5 in your book.

Solution 1

(a) Think about the row picture: if $A = \begin{bmatrix} u^\mathsf{T} \\ v^\mathsf{T} \\ (u+v)^\mathsf{T} \end{bmatrix}$, then $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ becomes the three equations $u^\mathsf{T} x = 1, v^\mathsf{T} x = 0, (u^\mathsf{T} + v^\mathsf{T})x = 0$. The sum of the first two equations is inconsistent with the third equation.
(b) In order for the sum of the first two equations to equal the third, we must have $b_1 + b_2 = b_3$.
(c) Elimination takes row 3 of A into a row of zeroes.

Problem 2 *Monday 9/11*

Do Problem #9 from section 2.5 in your book.

Solution 2

If $B = P_{12}A$ (where P_{12} is the permutation matrix swapping the first two rows), then $B^{-1} = A^{-1}P_{12}^{-1}$. So B^{-1} exists, provided (i) A^{-1} exists (we're allowed to assume this) and (ii) P_{12}^{-1} exists (this is just P_{12}).

Problem 3 *Monday 9/11*

Do Problem #25 from section 2.5 in your book.

Solution 3

(A) *"Gaussian"* elimination on $[\,A \quad I\,] = [\,LDU \quad I\,]$ gives

$$[\,DU \quad L^{-1}\,] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{bmatrix},$$

but we *keep going* with *"Gauss-Jordan"* elimination by dividing each row by its pivot:

$$[\,U \quad D^{-1}L^{-1}\,] = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

and eliminating earlier entries:

$$[\,I \quad U^{-1}D^{-1}L^{-1}\,] = [\,I \quad A^{-1}\,] = \begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -3/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & 1/4 & -1/4 & 3/4 \end{bmatrix}.$$

You probably did your eliminations in a different order: that's fine! (I did it this way to match the steps with the factors of $A = LDU$.) Any way you do it, there's only one inverse of A:

$$A^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}.$$

(B) B isn't invertible! If we try elimination —

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3/2 & -3/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

— we get a *zero row*.

(Another way of seeing that B can't be inverted: $B\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$.

Why does this prove B has no inverse?)

Problem 4 *Monday 9/11*

Compute, by Gauss-Jordan elimination, a formula for the inverse of the 2-by-2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
What assumptions did you need to make?

Solution 4

We have $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & (ad-bc)/a & -c/a & 1 \end{array}\right]$ (if $a \neq 0$)

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \text{ (if } a \neq 0, (ad-bc) \neq 0\text{)}$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}(1+\frac{bc}{ad-bc}) & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

so the inverse is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
We assumed both pivots (a and $ad-bc$) were nonzero.
The case $a = 0$ (and $ad - bc = -bc \neq 0$) has to be done separately, because we need to swap rows:

$$\left[\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right] \text{ (if } bc \neq 0\text{)}$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{0-bc} & \frac{-b}{0-bc} \\ 0 & 1 & \frac{-c}{0-bc} & \frac{0}{0-bc} \end{array}\right].$$

But we get the same formula for A^{-1}.

Problem 5 *Wednesday 9/13*

Do Problem #2 from section 2.6 in your book.

Solution 5

$\ell_{31} = 1, \ell_{32} = 2$.
Check that we can recover the original Row 3 from the elimination-reduced rows:

$$\begin{array}{rl} 1 & [1 \;\; 1 \;\; 1 \;\; 5] \\ +\;2 & [0 \;\; 1 \;\; 2 \;\; 2] \\ + & [0 \;\; 0 \;\; 1 \;\; 2] \\ \hline & [1 \;\; 3 \;\; 6 \;\; 11] \end{array}$$

Problem 6 *Wednesday 9/13*

(a) Do Problem #13 from section 2.6 in your book.
(b) Now compute the LDU-factorization $A = LDU$. Why is $U = L^{\mathsf{T}}$?

Solution 6

(a) We have $U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, provided $a \neq 0, b \neq a, c \neq b, d \neq c$.

(b) We have $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}$, and $U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Since A is symmetric, $U = L^{\mathsf{T}}$.

Problem 7 *Wednesday 9/13*

Do Problem #21 from section 2.6 in your book.

Solution 7

For A, we have $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \end{bmatrix}$ and $U = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$.

For B, we have $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & x & 1 & 0 \\ 0 & x & x & 1 \end{bmatrix}$ and $U = \begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$.

Problem 8 *Wednesday 9/13*

Back in 18.02, you learned a different method for solving $Ax = b$ called Cramer's rule, which wrote the coefficients of x as a quotient of determinants. You also learned to solve the system by finding A^{-1}.

Let's compare elimination, `x=inv(A)*b`, and Cramer's rule using Matlab.[1]

(a) Try each of the three methods on a random matrix of size 100, then of size 200 and 300. (You may want to try several matrices of each size, and take an average to get a better estimate.) What do you conclude?

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Some commands you may find useful:
>> A=rand(100);          % A is a random 100-by-100 matrix
>> b=rand(100,1);        % B is a random 100-by-1 column vector
>> tic; cramer(A,b); toc % Time the solution by Cramer's rule
>> tic; inv(A)*b; toc    % by matrix inversion
>> tic; A\b; toc         % by elimination
>> for i = 1:5
      ...
   end                   % do ... five times
```

(b) How much faster is elimination than Cramer or `inv(A)`? (You could also try larger sizes.)

Solution 8

Here's the data I collected for parts (a) and (b). (I ran each with five matrices, and reported the median. Times in seconds.)

[1] If you're not using Athena, you may need the teaching code `cramer.m`, available from the 18.06 Web page: `web.mit.edu/18.06/www`

n	Cramer	Inverse	Elimination	ratio I/E
100	0.486	0.0057	0.0036	1.58
200	5.12	0.031	0.016	1.94
300	21.1	0.091	0.048	1.90
400	68.6	0.21	0.12	1.75
500	-	0.37	0.18	2.05
1000	-	2.5	1.06	2.36
1500	-	8.9	3.7	2.41
2000	-	19.4	6.8	2.85
2500	-	40.5	13.3	3.05
3000	-	63.1	21.0	3.01

Notice that Cramer's rule isn't very efficient at all, compared to either of the other methods. (In fact, the situation is even worse for our 18.02 methods — Matlab uses elimination to compute determinants and inverses, behind the scenes. Computing determinants (and inverses) the way you learned in 18.02 would take so long, for even a random 100-by-100 matrix, that the Earth would be consumed by the Sun before you finished![2])

The other two methods are both much better by comparison. Notice how the speed advantage of elimination over finding the inverse gradually improves to our theoretical factor of 3. (It's not obvious from just this data that this is true, but we could always try larger matrices.)

Problem 9 *Wednesday 9/13*

Factor the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 6 & 2 & 1 \end{bmatrix}$ as $A = LU$. (*Timesaving hint:* Look at problem 8 from Problem Set 1.)

Solution 9

We already did the elimination $E_{32}E_{31}E_{21}A = U$ in the last problem set; we just need to write it in the form $A = LU$.

U is the same, namely $\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$; and

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

So $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 10 *Friday 9/15*

Suppose $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$. Let $b = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

(a) Solve $Lc = b$ by back-substitution, then solve $Ur = c$ the same way for r.
Why does $x = r$ satisfy $Ax = b$?

[2] The determinant expands into $100! = 9.3 \times 10^{157}$ terms. By comparison, your computer can do only a few billion operations per second.

(b) Oops! I meant to write $PA = LU$, where L,U are as above and $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.
What vector solves $Ax = b$ now?

Solution 10

(a) $b = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ gives $c = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$ gives $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
Since $Ar = LUr = Lc = b$, we have $Ar = b$.
(b) Now we have $PAx = LUx = Pb = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$, so we do exactly the same steps as before, but now for Pb instead of b. This gives $c' = \begin{bmatrix} 1 \\ 6 \\ 12 \end{bmatrix}$ and $x = r' = \begin{bmatrix} 7/2 \\ 6 \\ 6 \end{bmatrix}$.

Problem 11 *Friday 9/15*

(a) When is the product of two symmetric matrices symmetric? Explain your answer.

Solution 11

(a) AB is symmetric if and only if $AB = (AB)^\top$, by definition. Keep going! So $AB = (AB)^\top = B^\top A^\top = BA$ (because A and B are symmetric). In other words, AB *is symmetric if and only if* A *and* B *commute.*

Problem 12 *Friday 9/15*

Let $P = P_{13}P_{26}P_{36}P_{14}P_{43}$, where the P_{ij} are permutation matrices of order 6.

(a) What is $P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$?
(b) What is P?

Solution 12

(a) You can write out all the matrices and multiply them, but there's an easier way: $P_{43}x$ swaps rows 3 and 4 of x leaving $[1 \ \ 2 \ \ 4 \ \ 3 \ \ 5 \ \ 6]^\top$, then P_{14} swaps rows 1 and 4, giving us $[3 \ \ 2 \ \ 4 \ \ 1 \ \ 5 \ \ 6]^\top$, and so forth... giving us $[6 \ \ 4 \ \ 3 \ \ 1 \ \ 5 \ \ 2]^\top$.
(b) The permutation matrix taking rows 1,2,4,3,5,6 to 6,4,3,1,5,2 is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 13 *Friday 9/15*

(a) What happens if we multiply a matrix A by the elimination matrix E_{ji} on the *right*?
(b) Solve by elimination: $y^\top \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} = [1 \ \ -4 \ \ 3]$.
(c) How could you convert this to a row-elimination problem?

Solution 13

(a) You do column elimination: subtract ℓ times column j from column i.
(b) If we do column elimination, the matrix (call it A) becomes

$$\begin{bmatrix} \boxed{2} & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{2} & 0 & 0 \\ -1 & \boxed{5/2} & 1 \\ 0 & -1 & 2 \end{bmatrix} (\ell = 1/2) \rightsquigarrow \begin{bmatrix} \boxed{2} & 0 & 0 \\ -1 & \boxed{5/2} & 0 \\ 0 & -1 & \boxed{12/5} \end{bmatrix} (\ell = 2/5)$$

Performing the same steps on $\begin{bmatrix} 1 & 4 & 3 \end{bmatrix}$ yields

$$\begin{bmatrix} 1 & -4 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -9/2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -9/2 & 24/5 \end{bmatrix}.$$

Now we back-substitute: just to clarify, our equations are

$$\begin{aligned} 2y_1 - y_2 &= 1 \\ \frac{5}{2}y_2 - y_3 &= \frac{-9}{2} \\ \frac{12}{5}y_3 &= \frac{24}{5} \end{aligned}$$

so $y_3 = 2$, $y_2 = -1$, and $y_1 = 0$.
(c) Take the transpose of both sides to get $A^\mathsf{T} y = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.