# 18.06 Problem Set 10

Due Wednesday, Nov. 29, 2006 at **4:00 p.m.** in 2-106

**Problem 1** *Monday 11/20*

Do Problem #7 from section 8.1 in your book.

***Solution 1***

For five springs,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix}, K = A^{\mathrm{T}}CA = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4+c_5 \end{bmatrix}.$$

If $C = I$, then $K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$ and $Ku = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ has solution $u = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$.

**Problem 2** *Wednesday, 11/22*

Do Problem #3 from section 6.6 in your book.

***Solution 2***

$$\begin{aligned} B &= M^{-1}AM : \\ (i) \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ (ii) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ (iii) \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

**Problem 3** *Wednesday, 11/22*

Do Problem #11 from section 6.6 in your book.

***Solution 3***

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}' = J \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$ is the system of four equations

$$\begin{aligned} dw/dt &= 5w + x \\ dx/dt &= 5x + y \\ dy/dt &= 5y + z \\ dz/dt &= 5z \end{aligned}$$

The last is easy: $z = z(0)e^{5t}$. Now put that into the next equation: $dy/dt = 5y + z(0)e^{5t}$.
Solve this (it's first-order linear) to get $y = (y(0) + tz(0))e^{5t}$.
Now put that into the next equation and solve: $x = (x(0) + ty(0) + t^2 z(0))e^{5t}$.
And finally, the one we're looking for: $w = (w(0) + tx(0) + t^2 y(0) + t^3 z(0))e^{5t}$.

**Problem 4** *Wednesday, 11/22*

Do Problem #12 from section 6.6 in your book.

***Solution 4***

$M$ has to be a 4-by-4 matrix for $JM$ and $MK$ to even make sense. Suppose it looks like this:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

Then the products $JM$ and $MK$ look like this:

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \stackrel{?}{=} MK = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

Now compare the entries. In the third row $m_{31} = 0$, $m_{32} = m_{31}(= 0)$, $m_{33} = m_{32}(= 0)$, and $m_{34} = 0$. So $M$'s entire third row is zero — it doesn't have full row rank, so it can't have an inverse (and we can't have "$M^{-1}JM = K$" without $M^{-1}$!)

**Problem 5** *Monday, 11/27*

Do Problem #7 from section 6.7 in your book.

***Solution 5***

$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}(\lambda_1 = 3);\ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}(\lambda_2 = 1).\ v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}(\lambda_3 = 0);$
Take these orthonormal eigenvectors as the columns of $V$:
$V = [\, v_1 \quad v_2 \quad v_3 \,] = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$
and the eigenvalues $\lambda_i$ give us the singular values $\sigma_i = \sqrt{\lambda_i}$:
$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix}$

$AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(\lambda_1 = 3);\ u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}(\lambda_2 = 1).$
These are a set of orthonormal eigenvectors, but we can't write down $U$ just yet!
The problem is that there is not a unique choice of orthogonal eigenvectors ($u_1$ and $-u_1$ are both eigenvectors for $\lambda_1$, for example), and our choice for $U$ needs to match our choice for $V$ so we'll have $A = U\Sigma V^{\mathrm{T}}$.
If $A = U\Sigma V^{\mathrm{T}}$, then $AV = U\Sigma$, that is, $Av_i = \sigma_i u_i$, so we can solve for $u_i$:
$U = [\, u_1 \quad u_2 \,]$ (we got lucky!) $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Then we can check, by multiplying, that

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}}_{V^{\mathrm{T}}}$$

**Problem 6** *Monday, 11/27*

Do Problem #12 from section 6.7 in your book.

***Solution 6***

$A$ is symmetric, so $u_1$ and $u_2$ are orthogonal. Unfortunately, we can't just use the diagonalization $A = U\Lambda U^{\mathrm{T}}$ of $A$, because $\lambda_2$ is negative, and can't be a singular value. We can fix this by changing one of the two occurrences of $u_2$ to $-u_2$; then

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & \\ & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

**Problem 7** *Monday, 11/27*

Do Problem #15 from section 6.7 in your book.

***Solution 7***

$$\begin{aligned}
A &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} \\
(a)\quad 4A &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} \\
(b)\quad A^{\mathrm{T}} &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{\mathrm{T}} \\
A^{-1} &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 & \\ & 1/\sigma_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{\mathrm{T}}
\end{aligned}$$

**Problem 8** *for Wednesday, 11/29*

One way of thinking about matrix multiplication is as a *linear transformation* — just as $y = ax$ is a linear function transforming an input $x$ to an output $y$, we can think of $y = Ax$ as a "linear" function, transforming our input vector $x$ (in $R^n$) to output vector $y$ (in $R^m$).
Formally, a function $y = T(x)$ is "linear" if

- $T(u + v) = T(u) + T(v)$ (we can break up sums)
- $T(cv) = cT(v)$ (we can pull out constant factors)

So, for example, $f(x) = 3x$ is linear, because $3(x_1 + x_2) = 3x_1 + 3x_2$ and $3(cx) = c \cdot 3x$; but $f(x) = \sin(x)$ isn't linear, because $\sin(a + b) \neq \sin(a) + \sin(b)$.
Which of these are linear? Why or why not?

1. $y = x^2$ (input and output are in $R$)
2. $g(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 + 3x_2$ (input in $R^2$, output in $R$)
3. $f(x) = 3x + 1$ (careful!)
4. $T(x) = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} x$ (input and output are in $R^2$)
5. $L(f) = \int_0^1 f(t)\, dt$ (input is in a function space, output is in $R$)
6. $M(f) = \frac{d^2 f}{dt^2}$ (input and output are both functions)

***Solution 8***

1. Not linear: $(u + v)^2 \neq u^2 + v^2$.
2. Linear: $g(x + y) = g(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}) = (x_1 + y_1) + 3(x_2 + y_2)$ is the same as $g(x) + g(y) = (x_1 + 3x_2) + (y_1 + 3y_2)$, and $g(cx) = cx_1 + 3cx_2$ is the same as $cg(x) = c(x_1 + 3x_2)$.
3. Not linear: $3(x + y) + 1 \neq (3x + 1) + (3y + 1)$. *(Unlike your high-school algebra teacher, we* don't *consider $y = 3x + 1$ "linear"! Only equations like $y = Ax$ have the nice properties above. (You can call the more general equation $y = Ax + b$ an "affine transformation".))*
4. Linear: $T(x+y) = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ is the same as $T(x)+T(y) = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $T(cx) = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} = \begin{bmatrix} cx_1 + 2cx_2 \\ -cx_1 + 2cx_2 \end{bmatrix}$ is the same as $cT(x) = c\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c\begin{bmatrix} x_1 + 2x_2 \\ -x_1 + 2x_2 \end{bmatrix}$.
5. Linear: $L(f + g) = \int_0^1 f(t) + g(t)\, dt$ is the same as $L(f) + L(g) = \int_0^1 f(t)\, dt + \int_0^1 g(t)\, dt$, and $L(cf) = \int_0^1 cf(t)\, dt$ is the same as $cL(f) = c\int_0^1 f(t)\, dt$.
6. Linear: $M(f + g) = (f + g)''$ is the same as $M(f) + M(g) = f'' + g''$, and $M(cf) = (cf)''$ is the same as $cM(f) = cf''$.

**Problem 9** *for Wednesday, 11/29*

*If we pick a basis for the input and the output, we can write a linear transformation as a matrix.*
If $T(u) = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$ and $T(v) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, then what is $T(au + bv)$? Now write down a matrix $A$ for which $A\begin{bmatrix} a \\ b \end{bmatrix} = T(au + bv)$.

***Solution 9***

$T(au + bv) = \begin{bmatrix} 3a + b \\ -2a \\ -5a + 2b \end{bmatrix}$.

The matrix taking $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} 3a + b \\ -2a \\ -5a + 2b \end{bmatrix}$ is $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ -5 & 2 \end{bmatrix}$.