18.06 Problem Set 5

SOLUTIONS TO SELECTED PROBLEMS

1. Section 4.2, Problem 5

Answers: $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$; $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$; P_1P_2 is the zero matrix because $a_1 \perp a_2$ (projecting a vector onto a_2 and then projecting the result onto a_1 gives 0).

2. Section 4.2, Problem 13

Solution: The column space of A is the "xyz-hyperplane" in the 4-dimensional space, so the projection of $b = (1, 2, 3, 4)$ is $(1, 2, 3, 0)$. The projection matrix is $P = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$; it is a 4 by 4 matrix.

3. Section 4.2, Problem 17

Solution: We compute: $(I-P)^2 = I^2 - IP - PI + P^2 = I - P - P + P = I - P$. If P projects onto the column space of A, then $(I-P)v = v - Pv$ is the difference between a vector v and its projection onto the column space of A, which is the projection of v onto the space $A^\perp$, or the left nullspace of A.

4. Section 4.2, Problem 19

Answers: For example, we can choose $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$. (Of course, P will be the same for any choice of A.)

5. Section 4.2, Problem 27

Solution: If $A^TAx = 0$, then $Ax = 0$. The vector Ax is in the nullspace of A^T, and Ax is always in the column space of A. Since the nullspace of A^T is orthogonal to the column space of A, in order to be in both of these spaces Ax has to be 0.

6. Section 4.3, Problem 1

Answers: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^T A\hat{x} = A^T b$ gives $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$. Then $E = ||e||^2 = 44$.

7. Section 4.3, Problem 17

Answers: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\hat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

8. Section 4.3, Problem 22

Answer: The least squares equation is $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. Solution: $C = 1, D = -1$.

9. Section 4.3, Problem 26

Solution: Equating the slopes of the line connecting (t_1, b_1) and (t_2, b_2) and the line connecting (t_2, b_2) and (t_3, b_3), we get the equation $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$. Another way to state the condition of (t_i, b_i) being on one line is saying that (b_1, b_2, b_3) is in the column space of the matrix $A = \begin{bmatrix} 1 & t_1 \\ 2 & t_2 \\ 3 & t_3 \end{bmatrix}$. Equivalently, (b_1, b_2, b_3) is orthogonal to the complimentary space to the column space of A, which is spanned by the single vector $y = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$. Writing this condition algebraically, we get $(b_1, b_2, b_3) \cdot (t_2 - t_3, t_3 - t_1, t_1 - t_2) = 0$, or $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$ — essentially the same equation we got by equating slopes.

10. Section 4.3, Problem 27

Solution: The unsolvable system is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$. Then $A^T A =$

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$. Solving $A^T A\hat{x} = A^T b$ yields $\hat{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}$.
At $(x, y) = (0, 0)$, the plane $z = 2 - \frac{3}{2}x - \frac{3}{2}y$ has height 2, which is the average of 0,1,3,4.

11. Section 4.4, Problem 6

Solution: Unfortunately, the statement in the problem is false. The claim is true for *orthonormal* matrices: indeed, if $Q_1^T Q_1 = I$ and $Q_2^T Q_2 = I$, then $(Q_1 Q_2)^T Q_1 Q_2 = Q_2^T (Q_1^T Q_1) Q_2 = Q_2^T Q_2 = I$. Note that if Q_1 is orthogonal and Q_2 is orthonormal, then $Q_1 Q_2$ is orthogonal.

12. Section 4.4, Problem 7

Solution: The least squares solution is the solution to $Q^T Q\hat{x} = Q^T b$, which in this case reduces to $\hat{x} = Q^T b$.

13. (a) Minimize

$$\int_0^1 (c + dt - t^2)^2 dt = \int_0^1 (c^2 + 2cd + d^2 t - 2ct^2 - 2dt^3 + t^4) dt$$
$$= c^2 + cd + \frac{1}{3}d^2 - \frac{2}{3}c - \frac{2}{4}d + \frac{1}{5}$$

$$\begin{array}{ll} c\text{-derivative:} & 2c + d = \frac{2}{3} \\ d\text{-derivative:} & c + \frac{2}{3}d = \frac{2}{4} \end{array}$$

Solution: $c = -\frac{1}{6}$ and $d = 1$: Best line $y = t - \frac{1}{6}$.

Note: Dividing by 2 shows the 2 by 2 Hilbert matrix with $h_{ij} = 1/(i + j - 1)$:

$$\texttt{hilb}(2) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

(b) The 10 by 2 matrix is $A = \left[\texttt{ones}(10, 1)(1 : 10)'/10\right]$ and the column vector is $b = (1 : 10)'. * (1 : 10)'/100$.

$$A^{\mathrm{T}} A \begin{bmatrix} C \\ D \end{bmatrix} = A^{\mathrm{T}} b \quad \text{is} \quad \begin{bmatrix} 10 & 5.5 \\ 5.5 & 3.85 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 3.85 \\ 3.02 \end{bmatrix}$$

$$\text{giving} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -.22 \\ 1.1 \end{bmatrix}.$$

(c) The same calculation with 10 changed to 20 (and 100 to 400) comes closer to $c = -\frac{1}{6}, d = 1$:

$$A^{\mathrm{T}}A\begin{bmatrix} C \\ D \end{bmatrix} = A^{\mathrm{T}}b \quad \text{is} \quad \begin{bmatrix} 20 & 10.5 \\ 10.5 & 7.175 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7.175 \\ 5.5125 \end{bmatrix}$$

$$\text{giving} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -.1925 \\ 1.0500 \end{bmatrix}.$$

The error in comparing D to $d = 1$ dropped from .1 to .05 (exactly in half). The error in comparing C to $c = -\frac{1}{6}$ dropped from $c - C = .0533$ to $c - C = .0258$ (nearly in half).